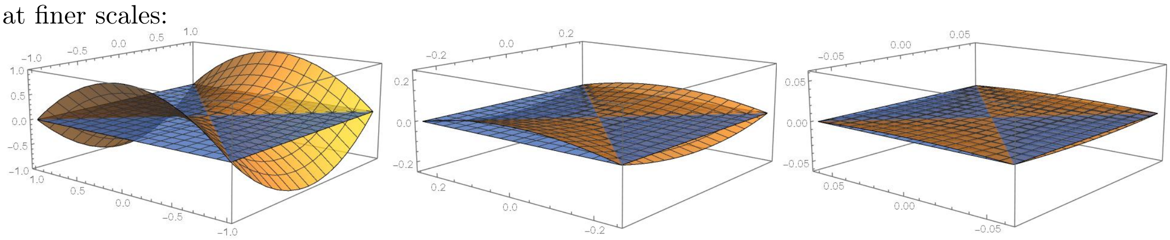
MAT 203 LECTURE OUTLINE 9/20

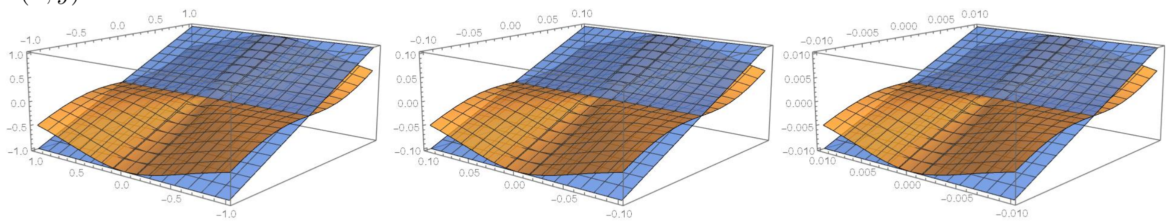
- Recall the definition of *derivative* for calculus of one variable,
$$f'(x) = \frac{df}{dx}(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$
Note the two main notations for the derivative, each with its own advantages. Also recall its interpretation as the slope of the line tangent to the graph of f at the point x. The statement that $f'(x)$ exists can be written as the statement that
$$f(x + \Delta x) = f(x) + f'(x)\Delta x + R(\Delta x),$$
where $R(\Delta x)/\Delta x \to 0$ as $\Delta x \to 0$. (R stands for "remainder".) To say this yet another way, the linear function
$$L_a(x) = f(a) + f'(a)(x - a)$$
(called the *linearization* or *linear approximation* of f at a) is a good approximation of $f(x)$ near the point a, provide that f is differentiable there.
- The most basic way to adapt this to multivariable functions is to define the *partial derivative* with respect to each variable. For a function $f(x, y)$ of two variables, the idea is to hold one variable fixed and treat f as a function of the other variable and take the derivative with respect to this variable. We have
$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$
$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$
This definition adopts to the case of $\mathbb{R}^3$ or $\mathbb{R}^n$ in the expected way.
- We can also do multiple partial derivatives. For example, the second-order partial derivatives are denoted by $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$. Note that $f_{xx} = (f_x)_x$ and so forth.
- Example. Find the first- and second-order partial derivatives of $f(x, y, z) = ye^x + x \ln z$.
- In the previous example, you likely noticed that $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$. In fact, this "equality of mixed partial derivaties" is always valid for reasonably nice functions. Specifically, it holds <u>whenever the second-order partial derivatives are continuous</u>. For those who are curious about a counterexample in the general case, the function
$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = 0 \end{cases}$$
has second-order partial derivatives f_{xy} and f_{yx} that are defined but not equal (and necessarily are not continuous).
- Recall the notion of differential in one-variable calculus: if $y = f(x)$, then we define $dy = f'(x)dx$. Likewise, for $z = f(x, y)$, we define the differential
$$dz = f_x(x, y)dx + f_y(x, y)dy,$$
where dx (resp. dy) represents a "small change in x (resp. y)". The point of this definition is that dz is (ideally) a good approximation of $f(x + \Delta x, y + \Delta y) - f(x, y)$ whenever Δx and Δy are small, with Δx being substituted in for dx and Δy for dy.
- <u>If the first partial derivatives f_x and f_y are continuous</u>, then the differential does indeed give a good <u>approximation for $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$</u>. More precisely, it holds that
$$\Delta z = f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$
for some $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. A function $f(x, y)$ for which this conclusion is true is said to be *differentiable*. If f_x and f_y are not continuous, then this conclusion may be false. An example to illustrate this is
$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = 0 \end{cases}.$$

We can compute the partial derivatives to get $f_x(0,0) = 1$ and $f_y(0,0) = 0$. Thus the linearization at $(0,0)$, if it were to exist, would be $L(x,y) = 1 \cdot (x-0) + 0 \cdot (y-0) = x$. However, $L(x,y)$ does not give a good approximation of $f(x,y)$ at the origin.

- It is helpful to try graphing some of these functions.
  - The "good" (that is, differentiable) function $f(x,y) = x^2 - y^2$ and its linearization $L(x,y) = 0$ at finer scales:

    Observe how $f(x,y)$ and $L(x,y)$ become practically indistinguishable as we zoom in.
  - The "bad" (that is, non-differentiable) function $f(x,y) = x^3/(x^2+y^2)$ and its linearization $L(x,y) = x$ at finer scales:

As we zoom in, the shape of the graph stays the same (can you justify this algebraically from the formula?) and so the linearization $L(x,y) = x$ never looks like the original function $f(x,y)$.