MAT 203 LECTURE OUTLINE 9/15

- Warmup: For a particle in motion:
  - Recall the formulas for $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, $a_{\mathbf{N}}$ and K.
  - If $a_{\mathbf{T}} = 0$, what does that tell you about the motion of the particle?
  - If $a_{\mathbf{N}} = 0$, what does that tell you about the motion of the particle?
  - If $K = 0$, what does that tell you about the motion of the particle?
- We are now beginning chapter 13 of the book. Here, the focus is on **differential** calculus of multivariable functions, i.e. functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ or $f\colon \mathbb{R}^3 \to \mathbb{R}$. Today we will get familiar with graphing multivariable functions and examining limits and continuity.
- Let's start with some examples. The variables x, y might represent location on the earth's surface. Surface air temperature or air pressure is then a function of the position (x, y).
- The most basic way to visualize a function is to let the z coordinate represent the value of the function: $z = f(x, y)$. We can then graph the function: those points $(x, y, z) \in \mathbb{R}^3$ for which $z = f(x, y)$.
- Another way to visualize a function is through *level sets* or *level curves* (also called *contour lines*). This is familiar from contour maps such as for elevation. Contour maps are also common in weather forecasting, where the function could be temperature or air pressure. This method has the advantage that a function of two variables can be represented on a two-dimensional graph.
- For functions of three variables, one cannot sketch a graph of the function in 3-dimensional space. However, one can sketch the level sets, which are typically surfaces.
- The definition of *limit* for multivariable functions is similar to the definition for single-variable functions, but there is some nuance to it. Recall the "δ-ε definition" of the limit from calculus of one variable: the statement $\lim_{x\to a} f(x) = L$ means: for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. This is a formal (that is, logically precise) definition that captures the idea that $f(x)$ becomes arbitrarily close to L when x is sufficiently close to a.
- Now we get to the definition of limit in multivariable calculus. Suppose that the function f is defined on an open disk centered at (x_0, y_0), except possibly at (x_0, y_0). The limit of the function f at (x_0, y_0) is L, and we write
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$
if the following condition is satisfied:
For each $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|f(x, y) - L| < \varepsilon$$
whenever
$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$
- For functions of a single variable, x can approach x_0 in one of two directions: from the right or from the left. The limit $\lim_{x\to x_0} f(x)$ exists if and only if both the left- and right-handed limits exist and are equal.
  For the limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ to exist, (x, y) must be able to approach (x_0, y_0) along any path and $f(x, y)$ should approach the same value L.
- We'll look at some examples to get an intuition for this. Each of these functions is defined except at the origin $(0, 0)$.
$$f_1(x, y) = \frac{1}{x^2 + y^2},$$
$$f_2(x, y) = \frac{x^2 y}{x^2 + y^2},$$
$$f_3(x, y) = \frac{xy}{x^2 + y^2},$$
$$f_4(x, y) = \frac{x^2 y^2}{x^2 + y^2}.$$
For which functions f_j does $\lim_{(x,y)\to(0,0)} f_j(x, y)$ exist, and what is the limit? This question can be done directly from the definition of limit.

- For f_1, the limit does not exist because f_1 is unbounded near $(0,0)$.
- For f_2, the limit does exist and is equal to zero. We can verify this from the definition: Observe that
$$|f_2(x,y)-0| = \left|\frac{x^2}{x^2+y^2}\right| \cdot |y| \leq 1 \cdot \sqrt{x^2+y^2}.$$
So $|f_2(x,y)-0| < \delta$ whenever $\sqrt{x^2+y^2} = \sqrt{(x-0)^2+(y-0)^2} < \delta$. Thus, given $\varepsilon > 0$, we choose $\delta = \varepsilon$. Thus, from the definition of limit we see that $\lim_{(x,y)\to(0,0)} f_2(x,y) = 0$. Take some time to really understand how we've used the definition of limit in this argument.
- For f_3, the limit does not exist. To see this, we can let (x,y) approach $(0,0)$ along two different paths. First, we have $\lim_{t\to 0} f_2(t,0) = \lim_{t\to 0} 0 = 0$. Second, we have $\lim_{t\to 0} f_2(t,t) = \lim_{t\to 0} \frac{tt}{t^2+t^2} = 1/2$. Since $0 \neq 1/2$, we conclude that the limit does not exist.
- For f_4, the limit does exist and is equal to zero. We illustrate another way to argue this without appealing directly to the definition: by using polar coordinates. In polar coordinates, the limit can be written as
$$\lim_{r\to 0} f_4(r,\theta) = \lim_{r\to 0} \frac{r^2\cos^2(\theta) r^2 \sin^2(\theta)}{r^2\sin(\theta) + r^2\cos(\theta)} = \lim_{r\to 0} r^2 \frac{\cos(\theta)\sin(\theta)}{1}.$$
Since $\cos(\theta)\sin(\theta)$ is continuous for all $\theta \in \mathbb{R}$ (in fact, $|\cos(\theta)\sin(\theta)| \leq 1$ for all $\theta \in \mathbb{R}$) and $r^2 \to 0$ as $r \to 0$, we see that $\lim_{r\to 0} f_4(r,\theta) = 0$.

- A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is *continuous* at (x_0,y_0) if $f(x_0,y_0) = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$.
  If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists but is different from $f(x_0,y_0)$, then f has a *removable discontinuity* at (x_0,y_0).
  Otherwise, f has a *non-removable discontinuity* at (x_0,y_0).
- Continuity is preserved under the standard operations:
  If $f,g\colon \mathbb{R}^2 \to \mathbb{R}$ are continuous at (x_0,y_0), then so are $f+g$, $f-g$ and fg. Also, f/g is continuous at (x_0,y_0) provided $g(x_0,y_0) \neq 0$.