MAT 203 LECTURE OUTLINE 9/1

- Last time, we looked at the problem of finding the distance between a given point and a given plane. Today, we begin with one more typical geometry problem in three-dimensional space: to find the distance between a given point Q and a given line L. What's interesting is that, while this is essentially a problem in trigonometry, the solution involves only the basic operations of addition and multiplication (via the dot/cross product) and square roots, not the trigonometric functions sin, cos, tan.
- Again, we have a simple method for doing this. First, pick any convenient point P belonging to the line L. Let $\mathbf{v}$ be the direction vector for the line L. Then the distance from Q to L is given by $\|\overrightarrow{PQ} \times \mathbf{v}\|/\|\mathbf{v}\|$. the magnitude of the cross product of $\overrightarrow{PQ}$ and $\mathbf{v}$. Try sketching a diagram to convince yourself that this formula is valid.
- Try this problem with the point $Q(3, -1, 4)$ and the line L defined by the equations $(x, y, z) = (-2 + 3t, -2t, 1 + 4t)$. The answer is $\sqrt{6}$.
- Section 11.6 covers three different types of surfaces in $\mathbb{R}^3$: cylindrical surfaces, quadric surfaces, and surfaces of revolution. We will focus on quadric surfaces.
- For reference, a *cylindrical surface* is one formed by taking a curve in a plane and forming a surface by taking translations of this curve in a fixed direction. The best known example is the right circular cylinder, which we often simply call a "cylinder". A *surface of rotation* is one formed by rotating a curve in a plane around a fixed axis passing through that plane.
- A *quadric surface* is one defined by a second-degree equation in x, y, z. (*second-degree* means each term has at most two variables in it, counting repeats, e.g., $2x^2 + 3yz - x = 0$ is a second-degree equation. A linear equation is a first-degree equation.) The general second-degree equation has the form
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J, .$$
where A, B, C, D, E, F are not all zero. Depending on the choice of $A, B, C, D, E, F, G, H, I, J$, we can get one of several basic types of surfaces.
- In this class, we will always assume that $D = E = F = 0$. That is, the defining equation for a quadric surface has no mixed terms. It turns out that you can always apply a linear change of variables (i.e., replace (x, y, z) with specially chosen variables $(u, v, w) = (f(x, y, z), g(x, y, z), h(x, y, z))$ where f, g, h are linear functions) to eliminate any mixed terms. This is an application of the idea of *diagonalizing* a matrix in linear algebra and so is beyond the scope of this course.
- A good method to identify a quadric surface is to graph the intersection of the surface with the xy-plane, yz-plane, and xz-plane. The intersection of the surface with a plane is called the *trace* of that surface with the plane. Each trace of a quadric surface is an ellipse, a hyperbola, a parabola, the union of two lines, or the empty set. How to recognize these is a precalculus topic (conic sections) you should have learned at some point. To recall briefly:
  — ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
  — hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
  — parabola: $y = \dfrac{x^2}{a^2}$
  — union of two lines: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$
  The different combinations of these different traces is what produces different quadric surfaces. There are six types, listed here with a representative equation:
  — ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
  — hyperboloid of one sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
  — hyperboloid of two sheets: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
  — elliptic cone: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$

— elliptic paraboloid: $z + \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

— hyperbolic paraboloid: $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

See the chart in the textbook for illustrations (note that there is a typo where "elliptic cone" appears in place of "ellipsoid").

- Sketch and identify the following equations (see problems 11.6.5-10 in the textbook): $x^2/9+y^2/16+z^2/9 = 1$; $4x^2 - y^2 + 4z^2 = 4$; $4x^2 - 4y + z^2 = 0$.
- There are two other common coordinate systems for $\mathbb{R}^3$ besides the standard rectangular coordinates (x, y, z): cylindrical coordinates and spherical coordinates. Recall that, in the plane, a point can be described by its radius (distance to the origin) and a single angle (measured from the positive x-axis).
- In cylindrical coordinates, a point P is represented by a triple (r, θ, z) satisfying the relations

$$x = r\cos(\theta),\ \ y = r\sin(\theta),\ \ z = z$$

and

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ \ z = z.$$

These relations indicate how to convert a point from rectangular coordinates to cylindrical and vice versa. Some care must be taken with finding θ for a given (x, y, z).

- In spherical coordinates, a point P is represented by a triple (ρ, θ, ϕ) satisfying the relations

$$x = \rho\sin(\phi)\cos(\theta),\ \ y = \rho\sin(\phi)\sin(\theta),\ \ z = \rho\cos(\phi)$$

and

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\theta) = \frac{y}{x},\ \cos(\phi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

- Convert $(x, y, z) = (3\sqrt{2}/2, 3\sqrt{2}/2, 1)$ to cylindrical coordinates. Convert $(x, y, z) = (-2, 2\sqrt{3}, 4)$ to spherical coordinates.
- In Chapter 12, we will study *vector-valued functions.* These are functions whose input is a single real number and whose output is a vector, here usually a 3-dimensional vector. These have the form $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. The image of such a function (for dimension three) is called a *space curve.* In the homework, you'll practice evaluating and graphing vector-valued functions.