MAT 203 LECTURE OUTLINE 8/30

- First, two examples to illustrate the ideas from last class:
- Example 1. Consider the points $P(0,0)$ and $Q(8,2)$, with the line segment from P to Q representing the surface of a ramp. (This is like Figure 11.28 in the textbook but with different numbers.) We place a box weighing 1000 lb. on this ramp that is acted on by gravity; assume that friction is negligible.
  (a) What is force of the ramp on the box?
  (b) What is the acceleration of the box? (The acceleration of an object in free fall is 32.17 ft./s^2)
- Side remark for those interested: this problem is complicated slightly since (following the textbook) we are using imperial units. In the metric system, there is a clear distinction between the kilogram as a unit of mass and the Newton as a unit of force. In the imperial system, a "pound" traditionally can represent either a mass or a weight (i.e., the force of gravity) since in past times these two quantities were not distinguished. There is a relatively new imperial unit called the "slug" that is meant to be the mass unit corresponding to a pound, but it is relatively obscure.
- In this problem we have two vectors: $\mathbf{u} = \langle 0, -1000\rangle$ representing the force of gravity and $\mathbf{v} = \langle 8, 2\rangle$ from the ramp.
- Let $\mathbf{w}_1$ be the projection of $\mathbf{u}$ onto $\mathbf{v}$ and $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ the orthogonal complement. Then the answer to (a) is $-\mathbf{w}_2$ since the ramp must counteract the force of gravity orthogonal to $\mathbf{v}$. The answer to (b) is closely related to $\mathbf{w}_1$; we just have to obtain the acceleration corresponding to the force $\mathbf{w}_1$.
- Use the formula $\mathbf{w}_1 = \mathbf{proj}_{\mathbf{v}}\,\mathbf{u}$ to get $\mathbf{w}_1 = \langle -4000/17, -1000/17\rangle$. Subtract to get $\mathbf{w}_2 = \langle 4000/17, -16000/17\rangle$. So the answer to (a) is $\langle -4000/17, 16000/17\rangle \approx \langle -235.3, 941.2\rangle$.
- To get (b), first we use the relation "force equals mass times acceleration" for an object in free-fall to get that the mass of the box is $1000/32.17 \approx 31.08$ slugs. Using the relation "force equals mass times acceleration" but applied to motion on the ramp, we have that the acceleration of the box is

$$\frac{\mathbf{w}_1}{1000/32.17} = \frac{32.17}{1000}\langle \frac{-4000}{17}, \frac{-1000}{17}\rangle \approx \langle -7.57, -1.89\rangle.$$

  If we're considered with the magnitude of the acceleration, this is approximately 7.8 ft./s^2.
- The second example relates to *torque*, or rotational force. Consider a rod with initial point P and terminal point Q; this rod rotates at the point P by a force $\mathbf{F}$ applied at point Q. Thus we have a position vector $\mathbf{u} = \overrightarrow{PQ}$ and a force vector $\mathbf{N}$.
- Torque has a magnitude and a direction. The magnitude satisfies the principle of the lever: force times the distance from the center of rotation. (This is an interesting physical principle: in theory, you can move the sun with your own arm strength simply by having a lever which is long enough. So what gives? The distance that you move the sun would be incredibly small.) The direction of the torque, **by convention**, is the direction orthogonal to the plane of rotation.
- The magnitude of torque also depends on how close the force $\mathbf{F}$ is to being orthogonal to $\mathbf{u}$. If $\mathbf{F}$ and $\mathbf{u}$ are nearly orthogonal, then the torque will be larger. In fact, the magnitude of the torque is the same as the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{F}$ (positioned with the same initial point). For these reasons, torque (denoted by τ) is defined using the cross product:

$$\tau = \mathbf{u} \times \mathbf{F}.$$

  Note the order of $\mathbf{u}$ and $\mathbf{F}$, since this determines the direction of τ according to the right-hand rule.
- Example 2. (See Example 11.3.4) Consider a rod of length 3 ft. at an angle of 60° from the horizontal plane being rotated by a downward force of 50 lbs. What is the resulting torque?
- Represent the rod by the vector $\mathbf{u}$ and the force by the vector $\mathbf{F}$. We'll fix a coordinate system. Let's say that the rod points in the direction of $\mathbf{j}$, so that $\mathbf{u} = 3\cos(60°)\mathbf{j} + 3\sin(60°)\mathbf{k} = \frac{3}{2}\mathbf{j} + \frac{3\sqrt{3}}{2}\mathbf{k}$ and $\mathbf{F} = -50\mathbf{k}$. Computing the cross product $\mathbf{u} \times \mathbf{F}$, we have $\tau = -75\mathbf{i}$
- The main topic for today is lines and planes in $\mathbb{R}^3$.
- Let's start with planes. Consider the equation $ax + by + cz + d = 0$, where $(x, y, z) \in \mathbb{R}^3$ is the unknown and a, b, c, d are constants with a, b, c not all zero. This is a single linear equation in three variables, hence the set of solutions has dimension $2 = 3 - 1$, hence a plane.

- A convenient way to sketch a plane (in the case where $d \neq 0$) is by finding its intersection with each coordinate axis (where possible), then connecting these points to form a triangle. See Figure 11.49 in the textbook. If there is no point of intersection with, for example, the x-axis, then the plane is parallel to the x-axis.
- A plane is uniquely determined by a point that is passes through and its normal vector $\mathbf{n}$. So how does $\mathbf{n}$ relate to the equation $ax + by + cy + d = 0$? The answer turns out to be quite simple: we may take $\mathbf{n} = \langle a, b, c \rangle$. (Of course, any non-zero scalar multiple of $\langle a, b, c \rangle$ is also a normal vector for the plane.) The proof of this is quite nice and may be found above Figure 11.45 in the textbook.
- Next, we look at lines. A line is uniquely determined by a point it passes through and a vector parallel to it. Let $P = P(x_0, y_0, z_0)$ and $\mathbf{v} = \langle a, b, c \rangle$ be a vector. Then the corresponding line is given by the equation
$$(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct),$$
where $t \in \mathbb{R}$ is a variable, called the *parameter*. These can also be written separately as
$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$$
- By solving each equation for t, we can represent the line by the system of equations
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$
(if $a, b, c \neq 0$). This is called *eliminating the parameter*. Essentially, this expresses the line as the solution to a system of 2 linear equations in 3 variables.
- One typical problem is to find the distance between a given point P and a given plane W. This means to find the shortest possible distance between P and some point in W. There is a simple method to solve this problem using the dot product. First, pick *any* point in W, ideally something as convenient as possible. Call it Q. Then find $\| \mathbf{proj_n} \overrightarrow{QP} \|$, the magnitude of the projection of $\overrightarrow{QP}$ onto the normal vector $\mathbf{n}$ for the plane. Recall that this is simply $|\overrightarrow{QP} \cdot \mathbf{n}| / \|\mathbf{n}\|$.
- Class problem 1. Find an equation for the plane containing the points $P(1, 0, 0)$, $Q(0, 2, 0)$, $R(0, 0, -2)$. [Note that this problem has multiple sets, including taking a cross product.]