MAT 203 LECTURE OUTLINE 11/3

- Today's class will go into more detail about the concept of conservative vector fields and how they relate to line integrals. Before going forward, review the definitions of vector field, conservative vector field, potential function, and line integral.
- **The Fundamental Theorem of Line Integrals.** If $\mathbf{F}(x, y, z) = \langle M, N, P\rangle$ is conservative in R, then
$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$
The proof of this theorem, done in class and found in the textbook, is achieved by reducing this theorem to the Fundamental Theorem of Calculus.
- The Fundamental Theorem for Line Integrals leads to the following combined theorem: Suppose $\mathbf{R} = \langle M, N, P\rangle$ has continuous first partial derivatives in an open connected set R and C is a piecewise smooth curve in R. Then the following are equivalent:
  (1) $\mathbf{F}$ is conservative: $\mathbf{F} = \nabla f$ for some f
  (2) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path
  (3) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C in R
- The moral is that we have options for how to evaluate line integrals of conservative vector fields. We can find the potential function explicitly, or we can use independence of path to choose the most convenient curve betweeen the endpoints.
- Example. Take $\mathbf{F}(x, y) = (y^3 + 1)\mathbf{i} + (3xy^2 + 1)\mathbf{j}$. Let C_1 be the upper semicircle from $(0, 0)$ to $(2, 0)$ centered at the point $(1, 0)$. Find $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

  We can do this integral directly. However, since $\mathbf{F}$ is conservative, two other ways are simpler. First, we can compute the potential function for $\mathbf{F}$ by integrating. This gives $f(x, y) = 3xy^2 + x + y$. So $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(2, 0) - f(0, 0) = 2$.

  Second, we can integrate over the straight-line path from $(0, 0)$ to $(2, 0)$. This reduces to $\int_0^2 1\, dt = 2$.
- In class, we did a few examples where you were asked to decide whether a vector field is conservative or not from looking at a plot of it.
- Recall that one way to check whether a vector field is conservative is by computing the curl and showing that the curl is 0. However, this only works when the vector field is defined on a simply connected region (this means that every close curve shrinks continuously to a point without leaving the region; more intuitively, the region doesn't have "holes"). For example, consider the vector field $\mathbf{F}(x, y) = \langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\rangle$. You can check by computation that curl $\mathbf{F} = 0$. But is $\mathbf{F}$ conservative?

  This depends on the domain R that we are considering in the problem. The issue is that $\mathbf{F}$ is not defined at the origin $(0, 0)$. If R is a domain that encircles the origin (for example, $R = \mathbb{R}^2 \setminus \{(0, 0)\}$), then $\mathbf{F}$ is not conservative. For example, take C to be the unit circle traversed once counterclockwise. Then $\displaystyle\int_X \mathbf{F} \cdot d\mathbf{r} = 2\pi \neq 0$ despite C being a closed curve. The idea is that the curve C can't shrink continuously to a point without crossing the origin $(0, 0)$, which is where $\mathbf{F}$ is not defined. You might say that $\mathbf{F}$ has a "pole" at the origin.

  On the other hand, if R is a region that doesn't encircle the origin (for example, $R = \{(x, y) : x > 0\}$, then this issue goes away and R is a conservative vector field.
- As a final subtopic, we'll look at the principle of Conservation of Energy. In this context, it means the following: For a particle moving in a conservative force field (and acted on only by the force field), the sum of its potential and kinetic energies is constant. The book attributes this law to Michael Faraday and calls it among the greatest laws in all of science.

- We need a few definitions to understand the preceding statement. The particle has position $\mathbf{r}(t)$ at time t, velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$. The force field $\mathbf{F}$ obeys Newton's second law: $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{a}(t)$, where m is the mass of a the particle. The potential function for $\mathbf{F}$ is denoted by f, and the potential energy of the particle at point A is $p(A) = -f(A)$. The kinetic energy at time t is $k(A) = \frac{1}{2}m\|\mathbf{v}\|^2$, where $A = \mathbf{r}(t)$.

  The proof of the law of Conservation of Energy can be found in the book. To summarize, the idea is to consider the particle moving on a path C from point $A = \mathbf{r}(a)$ to point $B = \mathbf{r}(b)$. We compute the work in two ways: using the potential function, which yields the difference in potential energy, and using the relation $\mathbf{F} = m\mathbf{a}$, which gives the difference in kinetic energy. These differences are the same, thus we get $p(A) + k(A) = p(B) + k(B)$.