MAT 203 LECTURE OUTLINE 11/1

- In Ch. 12, we studied vector-valued functions (of a single variable): functions $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^2$ or $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$. In Ch. 13-14, we studied multivariable functions: functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ or $f\colon \mathbb{R}^3 \to \mathbb{R}$. In this context, we can also call these scalar fields.
- In Ch. 15, we will study vector fields: functions $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$. Ch. 15 involves new ideas that you probably haven't encountered before in a math class (though maybe in a physics class)
- We have already seen one important example of a vector field: the gradient $\nabla f\colon \mathbb{R}^3 \to \mathbb{R}^3$ of a multivariable function.
- There are three especially important physical examples:
  - Velocity fields
  - Gravitational fields
  - Magnetic or electric fields

  The latter two have the form $\mathbf{F}(x, y, z) = \dfrac{-k}{\|\mathbf{r}\|^2}\left(\dfrac{\mathbf{r}}{\|\mathbf{r}\|}\right)$ for some $k > 0$, where $\mathbf{r} = \langle x, y, z\rangle$. These fields are said to follow an inverse square law.
- Both scalar fields and vector fields can be integrated over curves. This is traditionally called a *line integral*, even though more accurate terminology would be *curve integral*.
- Line integrals for scalar fields can be interpreted in the following way: think of the curve as a wire; the scalar field represents the density of the wire, the integral represents its total mass.
- Line integrals for vector fields can be interpreted in the following way: think of the curve as the path of a particle; the vector field represents a force field such as a gravitational field, the integral represents the work done by the force field on the particle.
- Now we get to the definitions. Let C be a piecewise smooth curve parametrized by the vector-valued function $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$. Let $f(x, y, z)$ be a scalar field. The *line integral of f over C* is denoted by $\displaystyle\int_C f\,ds$, where "$s$" is the variable for arc length. It is formally defined by a Riemann sum:

  $$\int_C f\,ds = \lim \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i,$$

  the limit taken over a sequence of partitions of the curve with mesh going to zero.
- Suppose C is parametrized by $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$, where $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. The integral of $f(x, y, z)$ can be evaluated by the formula

  $$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt.$$
- If $f(x, y, z) = 1$, then this is just the formula for arc length.
- Example. Let C be the straight line from the origin to $(1, 2, 1)$. Find $\displaystyle\int_C (x^2 - y + 3z)\,ds$.

  Note that part of the problem is finding a suitable parametrization $\mathbf{r}(t)$ for the curve C. The answer is $5\sqrt{6}/6$.
- Next, we give the definitions for line integrals of vector fields. Let C be a piecewise smooth curve parametrized by the vector-valued function $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$. Let $\mathbf{F}(x, y, z)$ be a vector field. Recall that $\mathbf{T}$ is the unit tangent vector for $\mathbf{r}$. The line integral of $\mathbf{F}$ over C is

  $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds.$$
- This is equal to

  $$\int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\,dt,$$

the form most convenient for computations. If $\mathbf{F} = \langle M, N, P\rangle$, this can be written equivalently as

$$\int_a^b M\,dx + N\,dy + P\,dz,$$

where dx gets replaced by $x'(t)dt$, and so forth, when evaluating.

- Example. Consider a gravitional field for a universe of two objects: object 1 at the origin (assume object 1 is much larger and its position is fixed) and object 2 at the point (x, y, z), which is variable. So $k = Gm_1m_2$, where G is the gravitational constant, m_i is the mass of object i. How much work does it take to move object 2 from the position $(1, 0, 0)$ in a straight line to the position $(4, 0, 0)$?

  We find the work done by gravity as a line integral of the gravitational field. We have $\mathbf{r}(t) = \langle t, 0, 0\rangle$, where $1 \leq t \leq 4$. The integral is

$$\int_1^4 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_1^4 \langle \frac{-kt}{(t^2+0+0)^{3/2}}, 0, 0\rangle \cdot \langle 1, 0, 0\rangle\,dt = \int_1^4 \frac{-k}{t^2} = -3k/4.$$

  Thus the work we must do to move object 2 is $3k/4$.

  A follow-up question is whether this answer changes if the particle moves along a different curve from $(1, 0, 0)$ to $(4, 0, 0)$. The answer is 'no', and try to convince yourself from a physical point of view why this is the case.

- Certain vector fields, such as gravitational fields, have the property of <u>path independence</u>: the value of line integrals depends only on the endpoints of the curve and not on the curve itself. Such a vector field is called conservative for reasons based in physics. This can be defined in the following way: a vector field $\mathbf{F}$ is <u>conservative</u> if there exists a differentiable function f such that $\mathbf{F} = \nabla f$.
- For example, we can compute the potential in the gravitational field example to be $f(x, y, z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}} = \dfrac{1}{\|\mathbf{r}\|}$.
- A way to test whether a vector field is conservative is the curl operation. The *curl* of a vector field $\mathbf{F} = \langle M, N, P\rangle$ is defined as

$$\begin{aligned} \text{curl } \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}. \end{aligned}$$

- Suppose that $\mathbf{F}$ is a vector field defined on a simply connected region in $\mathbb{R}^3$ (that is, every closed curve in the region can be shrunk to a point without leaving the region; for example, all of $\mathbb{R}^3$ or an open ball in $\mathbb{R}^3$). Then $\mathbf{F}$ is conservative if and only if curl $\mathbf{F}(x, y, z) = 0$ for all (x, y, z).
- A vector field $\mathbf{F}$ satisfying curl $\mathbf{F} = 0$ is called <u>irrotational</u>. If $\mathbf{F}$ is defined on a simply closed region, then this is the same thing as being conservative. If the region is not simply connected, then conservative implies irrotational (this reflects the fact that mixed partial derivatives commute: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ etc.) but not necessarily the other way around. Try computing the curl of the gravitational field example and verify that it is equal to zero.
- The other main vector operation is divergence. We'll discuss physical interpretations of curl and divergence another time. Let $\mathbf{F} = \langle M, N, P\rangle$ be a vector field. The divergence of $\mathbf{F}$ is

$$\text{div } \mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$