MAT 203 LECTURE OUTLINE 10/25

- The main topic of the day is triple integrals. The same ideas we used to define double integrals can also be used to define triple integrals. The triple integral of a function $f(x, y, z)$ over a region R in $\mathbb{R}^3$ is denoted by
$$\iiint_R f(x,y,z)\,dV.$$
- The triple integral of $f(x, y, z)$ over R is defined as the limit of a Riemann sum over <u>three-dimensional cubes</u>:
$$\iiint_R f(x,y,z)\,dV = \lim \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta x_i \Delta y_i \Delta z_i,$$
where in the limit the partitions become arbitrarily fine, provided this limit exists.
- Just as the double integral of $f(x, y)$ represents "volume under the surface $f(x, y)$", the triple integral of $f(x, y, z)$ can be thought of as the "four-dimensional volume contained under the three-dimensional graph of $f(x, y, z)$"
- However, this is probably hard to visualize. Instead, we can think of R as representing a solid object, $f(x, y, z)$ representing its density, and $\iiint_R f(x,y,z)\,dV$ representing the mass of this object.
- Fubini's theorem applies to triple integrals: a triple integral can be evaluated as three iterated integrals.
- Suppose that R is the region in $\mathbb{R}^3$ defined by
$$\begin{aligned} &a \le x \le b \\ &g_1(x) \le y \le g_2(x) \\ &h_1(x,y) \le z \le h_2(x,y). \end{aligned}$$
Then
$$\iiint_R f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$
Similar formulas hold for different orders of x, y, z.
- Now, let's move on to doing triple integrals in cylindrical coordinates. This is the direct extension of integration in polar coordinates covered last week.
- For example, suppose a region R is described in cylindrical coordinates by
$$\begin{aligned} &a \le \theta \le b \\ &g_1(\theta) \le r \le g_2(\theta) \\ &h_1(r,\theta) \le z \le h_2(r,\theta). \end{aligned}$$
Then
$$\iiint_R f(x,y,z)\,dV = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos(\theta), r\sin(\theta), z) r\,dz\,dr\,d\theta.$$
- There is also a form of integration adapted to spherical coordinates. For simplicity, suppose a region R is described in spherical coordinates by
$$\begin{aligned} &\rho_1 \le \rho \le \rho_2 \\ &\theta_1 \le \theta \le \theta_2 \\ &\phi_1 \le \phi \le \phi_2. \end{aligned}$$
Then
$$\iiint_R f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.$$

- Try computing the volume of the unit sphere in $\mathbb{R}^3$ in three ways: as a triple integral in rectangular coordinates, in cylindrical coordinates, and in spherical coordinates. Which one do you like best?

  For your reference, here are the three different setups:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx$$

$$\int_{0}^{2\pi}\int_{0}^{1}\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$$

$$\int_{0}^{\pi}\int_{0}^{2\pi}\int_{0}^{1} \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

  The answer is $4\pi/3$.