Midterm 3 Key

1. (a) $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable; $\mathbb{R}$ is not

(b) 17/99 (since $17 = .\overline{17} \cdot 100 - .\overline{17} = .\overline{17} \cdot 99$)

(c) $q = -2$ and $r = 4$ ($-10 = (-2) \cdot 7 + 4$)

(d) $132 = 54 \cdot 2 + 24$; $54 = 2 \cdot 24 + 6$; 6 divides 24. So $\gcd(132, 54) = 6$.

(e) Saturday ($365 \equiv 1 \mod 7$)

(f) $x = 5$. Note that $8 \equiv 35 \mod 9$. Since $\gcd(7, 9) = 1$, we can then divide by 7 to get $x \equiv 5 \mod 9$.

(g) $[1]_{12}, [5]_{12}, [7]_{12}, [11]_{12}$, since these numbers are relatively prime to 12.

2. See p. 178-179 in the textbook.

3. $x \equiv 14 \mod 41$. First, apply the Euclidean algorithm:

$$\begin{array}{rrl} & 164 = & 1 \cdot 164 + 0 \cdot 60 \\ (-2) & 60 = & 0 \cdot 164 + 1 \cdot 60 \\ (-1) & 44 = & 1 \cdot 164 + (-2) \cdot 60 \\ (-2) & 16 = & (-1) \cdot 164 + 3 \cdot 60 \\ (-1) & 12 = & 3 \cdot 164 + (-8) \cdot 60 \\ (-3) & 4 = & (-4) \cdot 164 + 11 \cdot 60 \end{array}$$

This shows that $\gcd(164, 60) = 4$, and (multiplying both sides by 5) gives the individual solution $x = 55$. Observe that $164/4 = 41$, and that $55 \equiv 14 \mod 41$. So the general solution is $x \equiv 14 \mod 41$, or $x = 14 + 41q$, $q \in \mathbb{Z}$.

4. See p. 237 in the textbook.