Name and ID: ....

Final Exam Logic, language and proof, MAT 200 Lec 01 Fall 2018
Dec 12 Wed 8:30-11:00 pm, Engineering 145
Two pages, 7 problems.

1. (a) Make the **truth table** of the formula $(P \leftrightarrow Q) \wedge (Q \rightarrow R)$.
   (4 points)

   (b) **Assign True and False** values to the variables P, Q, R such that the value of the formula $(P \wedge (Q \vee R)) \leftrightarrow ((P \wedge Q) \vee R)$ is **false**.
   (4 points)

   (c) **Negate** the formula $(P \leftrightarrow Q) \vee (R \rightarrow S)$. You are allowed to use the '$\neg$' sign in front of letters P, Q, R, S only.
   (4 points)

2. (a) **Negate** the following statement. Do not use words "no", "not" in your final answer. (Recall that for real numbers a, b the negation of $a > b$ is $a \leq b$.)
   There exists a real number x such that there exists a real number K such that for every real number R there exists a real number y such that $R < y$ and $f(x, y) > K$.
   (4 points)

   (b) Consider the set (interval) $I = \left(\frac{1}{2}, \frac{5}{2}\right] = \{r \in \mathbb{R} : \frac{1}{2} < r \leq \frac{5}{2}\}$. In each of the following cases, decide whether the statement is true or false. **Give a short justification for your answer.** Recall that $\mathbb{Z}$ is the set of all integers and $\mathbb{R}$ is the set of all real numbers.
   (6 points)

(a) $\forall y \in I : \exists x \in I : x < y$	(c) $\exists y \in I : \forall x \in I : x < y$	(e) $\exists k \in \mathbb{Z} : \forall u \in I : u \leq k$
(b) $\forall x \in I : \exists y \in I : x < y$	(d) $\exists y \in I : \forall x \in I : x \leq y$	(f) $\exists r \in \mathbb{R} : \forall u \in I : r < u$

   (c) Prove that for every positive real number $\varepsilon > 0$ there exists a positive real number t such that for every positive real number x such that $0 < x < t$ we have $\sqrt{x} < \varepsilon$.
   (6 points)

3. (a) Let x be a real number different from 1. Assume $0 < \frac{x^2}{x-1}$. Prove that $4 \leq \frac{x^2}{x-1}$. (Hint: first prove that $1 < x$.)
   (8 points)

   (b) Let m be an integer. Prove that $8m + 4$ is not a third power of any integer, that is, there is no $n \in \mathbb{Z}$ such that $n^3 = 8m + 4$.
   (8 points)

	1	2	3	4	5	6	7	**Total**
Max	*12 p*	*16 p*	*16 p*	*12 p*	*16 p*	*14 p*	*14 p*	***100 p***

THE EXAM CONTINUES ON A SECOND PAGE

4. Let A, B, C, D be four sets and consider

$$X = (A \setminus C) \times (B \cup D),$$

$$Y = ((A \cup C) \times (B \cup D)) \setminus ((A \cap C) \times (B \cap D)).$$

(a) Prove that $X \subseteq Y$.

(8 points)

(b) Let $A = \{1, 2\}, B = \{5, 6\}, C = \{2, 3\}, D = \{6, 7\}$. Enumerate all the elements of the set X and all the elements of the set Y (as defined above).

(4 points)

5. (a) Prove that for every positive integer n the number $2^{n+2} + 3^{2n+1}$ is divisible by 7.

(8 points)

(b) Let us define a sequence of real numbers recursively as $a_0 = \frac{2}{5}$ and for every positive integer j let $a_j = \frac{1}{2}\left(a_{j-1}^2 + a_{j-1} + \frac{1}{4}\right)$. Prove that for every non-negative integer n, we have

$$0 < a_n < \frac{1}{2}.$$

(8 points)

6. (a) Consider the interval $I = [1, 4] = \{r \in \mathbb{R} : 1 \leq r \leq 4\}$. In each case, decide whether the relation R on I is **reflexive, symmetric and transitive (three answers in each case)**. **Justify your answers.**

i. xR_1y iff $x - y < 2$;

ii. xR_2y iff $-2 < x - y < 2$;

iii. xR_3y iff $\exists k \in \mathbb{Z} : x - y = \frac{k}{2}$.

(10 points)

(b) Which one of the above relations is an equivalence relation? Enumerate all the elements from $I = [1, 4]$ equivalent to $\frac{5}{2}$ with respect to that equivalence relation.

(4 points)

7. (a) Let $f : X \to Y$ be a function, and $U_1 \subseteq X$, $U_2 \subseteq X$ two subsets of the domain.

i. Prove that $f(U_1) \setminus f(U_2) \subseteq f(U_1 \setminus U_2)$.

ii. Give an example of two sets X, Y, a function $f : X \to Y$ and two subsets $U_1 \subseteq X, U_2 \subseteq X$ such that $f(U_1) \setminus f(U_2) \neq f(U_1 \setminus U_2)$.

(8 points)

(b) i. Let X, Y, Z be three sets and $f : X \to Y$ and $g : Y \to Z$ two functions. Prove that if the composition $g \circ f : X \to Z$ is injective then the first function f also has to be injective.

ii. Give an example of three sets X, Y, Z and two functions $f : X \to Y$ and $g : Y \to Z$ such that the composition $g \circ f : X \to Z$ is injective but the second function g is not injective.

(6 points)