FINAL EXAM, MAT 200, LECTURE 02: FALL 2017

Instructions: Complete all 10 problems. The use of calculators, phones, books, or notes is not allowed during the exam time. Clearly mark and refer to any scratch work you want to be considered part of your answers. Each problem is worth 20 points and the exam is worth 200 points (i.e. 200 points is 100%, 100 points is an 50%, etc).

Please fill in the information asked for directly below.

Print Name:

ID Number:

1	2	3	4	5	6	7	8	9	10	Total (/200)

Problem 1: Show $2n^3 - 3n^2 + n$ is divisible by 6 for all $n \in \mathbb{Z}$.

Problem 2: Consider the function $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x, y) = x/(2y)$. Is f injective? Is f surjective? What is the image of f?

Problem 3: There are 100 residents in a town. They elect 10 residents to the city council. The council then elects 3 of its members to be officers. What is the number of possible outcomes of this election process?

Problem 4: Give a proof or a counterexample of the following statements

(1) $\forall r \in \mathbb{R}, \exists s \in \mathbb{R}$ s.t. $s^2 + 1 < r$
(2) $\exists x \in \mathbb{R}$ s.t. $\forall r \in \mathbb{R}, (r \in \mathbb{Q} \Rightarrow rx \notin \mathbb{Q})$
(3) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $x + y$ is even

Problem 5: Let a_n be a sequence defined by

$$a_1 = 1, \quad a_{n+1} = \frac{1}{2}(a_n + \frac{3}{a_n}).$$

Prove that, for all $n \geq 1$, we have $1 \leq a_n \leq 3$.

Problem 6: There are 80 students who are enrolled in analysis, algebra, or topology. Each class has exactly 40 students. 12 are taking only analysis, 13 are taking only algebra, and 14 are taking only topology. How many students are taking all three?

Hint: Recall that for three sets X, Y, and Z the inclusion-exclusion principle says

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|.$$

Problem 7: Use the Pigeonhole Principle to show that any collection of 5 different numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ contains a pair whose sum is 9.

Problem 8: Prove that the set of irrational real numbers is not denumerable. You may use the fact that the set of real numbers is not denumerable.

Problem 9: Consider the relation $\sim$ on $\mathbb{R}^2$ defined by: $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$. Is $\sim$ an equivalence relation? If so, describe the equivalence class $[(0, \frac{1}{2})]$.

Problem 10: Let A and B be finite sets with $|A| = m$ and $|B| = n$. For which pairs of whole numbers m and n is $|\mathcal{P}(A \times B)| = |\mathcal{P}(A) \times \mathcal{P}(B)|$? For every such pair, construct a bijection between $\mathcal{P}(A \times B)$ and $\mathcal{P}(A) \times \mathcal{P}(B)$.

(Scratch work)

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