ON CAT(κ) SURFACES

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Abstract. We study the properties of CAT(κ) surfaces: length metric spaces homeomorphic to a surface having curvature bounded above in the sense of satisfying the CAT(κ) condition locally. We provide three distinct proofs of the fact that CAT(κ) surfaces have bounded integral curvature. This fact allows us to use the established theory of surfaces of bounded curvature to derive further properties of CAT(κ) surfaces. Among other results, we show that such surfaces can be approximated by Riemannian surfaces of Gaussian curvature at most κ. We do this by giving explicit formulas for smoothing the vertices of model polyhedral surfaces.

1. Introduction

The field of Alexandrov geometry concerns two main classes of metric spaces: spaces of curvature bounded below and spaces of curvature bounded above. The latter is known at the class of CAT(κ) *spaces*, where $\kappa \in \mathbb{R}$ is some real parameter; these can be defined as spaces in which every sufficiently small geodesic triangle is thinner than the triangle of matching edge lengths in the model space of constant curvature κ. Alexandrov geometry provides a synthetic or coordinate-free approach to the intrinsic geometry of manifolds or more complicated spaces, generalizing ideas from the classical Riemannian geometry to potentially non-smooth spaces. CAT(κ) spaces were first studied systematically by Alexandrov [4] in the 1950s, building on his investigations on the intrinsic geometry of convex surfaces; see also earlier work of Busemann [13].

In this paper, we study the class of metric surfaces, i.e., metric spaces homeomorphic to a topological 2-manifold, satisfying the CAT(κ) condition locally, which we refer to more concisely as CAT(κ) *surfaces*. To our knowledge, the basic facts about CAT(κ) surfaces belong mainly to mathematical folklore. Nevertheless, there is continued interest in the topic; see for instance [1, 9, 14] for recent work involving CAT(κ) surfaces. We wish to take advantage of this situation to give the topic a thorough treatment.

1.1. CAT(κ) **and bounded integral curvature.** The fundamental fact about CAT(κ) surfaces is that they belong to another class of spaces also studied by Alexandrov and the Leningrad school of geometry associated with him, namely *surfaces of bounded (integral) curvature*. Roughly speaking, this is the largest class of surfaces for which curvature can be meaningfully defined as a signed Radon measure. Its precise definition involves the notion of *(angular) excess* of a geodesic triangle T, denoted by $\delta(T)$. A surface of bounded curvature is one for which, in every compactly contained neighborhood, there is a uniform upper bound on the sum of excesses of an arbitrary finite collection of non-overlapping simple triangles (see sections 2 and 3). There is a rich and well-developed theory of surfaces of bounded curvature which will help us derive further properties of CAT(κ) surfaces. We refer the reader to monographs by Alexandrov–Zalgaller [2] and Reshetnyak [37] as well as the surveys [18, 40]. We note as well that surfaces of curvature bounded below also have bounded curvature; see [38] for a proof. Our first objective is then the following theorem.

Theorem 1.1. *Let X be a* CAT(κ) *surface. Then X has bounded integral curvature.*

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Theorem 1.1 is trivial if $\kappa \leq 0$, since in this case every triangle has non-positive excess. In the case where $\kappa > 0$ its validity is *a priori* not clear. We are not aware of any complete proof of Theorem 1.1 in the literature, although a proof is sketched in Machigashira–Ohtsuka [30]; see also related discussion in [17]. In fact, Lemma 5.3 in [30] states, in more generality, the following basic inequality relating excess and curvature, from which Theorem 1.1 follows easily.

Theorem 1.2. *Let T be a triangle in a* $\mathrm{CAT}(\kappa)$ *surface. Then $\delta(T) \leq \kappa|T|$.*

Here, $|T|$ denotes the area of the interior of T, which we define as the Hausdorff 2-measure. According to [30], Theorem 1.2 can be established in a similar manner to an analogous inequality for surfaces with curvature bounded below, which was done in earlier work of Machigashira [29, Theorem 2.0]. However, there are various qualitative differences between $\mathrm{CAT}(\kappa)$ spaces and spaces of curvature bounded below, and thus we feel that it is not straightforward to adapt the proof in [29] to the $\mathrm{CAT}(\kappa)$ case.

In this paper, we give three distinct proofs of Theorem 1.1, all of which we consider to have their own points of interest. The first is a direct geometric proof, intended to establish Theorem 1.1 as efficiently as possible with a minimum of prerequisites (namely, Reshetnyak's majorization theorem and the Besicovitch inequality). This proof is based on proving a non-sharp version of the inequality in Theorem 1.2. Later in the paper, we show how Theorem 1.2 itself follows easily as a consequence of the theory of surfaces of bounded curvature. The other two proofs rely on a recent theorem of the third-named author with Creutz [16] on decomposing an arbitrary surface with a length metric into non-overlapping simple convex triangles, which we refer to as a *triangulation* of the the surface. The main tool for these proofs is a special type of triangulation that can be obtained by inductively subdividing a base triangulation, which we call a *vertex-edge triangulation.* In the second proof, we use this approach to verify the bounded curvature condition directly, while in the third proof we verify instead an equivalent definition of being a limit of surfaces of uniformly bounded curvature. We attribute the basic idea underlying the third proof to discussions with François Fillastre [19] and a MathOverflow post by Anton Petrunin [35]. Of the three proofs, we feel it is the second that is the most comprehensive and insightful.

One technical point relates to the definition of *area.* As stated above, we take the Hausdorff 2-measure as our definition. The original work of Alexandrov (cf. [4]) uses a different definition based on approximation by Euclidean polyhedral surfaces, which we call the *Alexandrov area.* We verify that the two notions of area are equivalent for $\mathrm{CAT}(\kappa)$ surfaces, or more generally for surfaces of bounded curvature without cusp points.

Proposition 1.3. *For any surface of bounded curvature without cusp points, the Alexandrov area coincides with the Hausdorff 2-measure.*

1.2. **Smooth approximation.** The next objective of this paper concerns approximation by smooth Riemannian surfaces. We prove the following theorem.

Theorem 1.4. *Let (X,d) be a* $\mathrm{CAT}(\kappa)$ *surface for some $\kappa \in \mathbb{R}$. Then there exists a sequence of smooth Riemannian metrics d_n on X with Gaussian curvature at most κ such that the sequence (X,d_n) converges uniformly to (X,d). Moreover, the metrics d_n have the property that $\limsup_{n\to\infty} |A_n| = |A|$ for all compact sets $A \subset X$. Here, A_n denotes the same set A equipped with the metric d_n.*

While one naturally expects Theorem 1.4 to be true, we have not found any statement in the literature, with the exception of a lemma of Labeni covering the case where $\kappa < 0$ [24]. Labeni's argument is based on the approximating cones with vertices of negative curvature in anti-de Sitter space by smooth convex surfaces. On the other hand, the corresponding statement for surfaces of curvature bounded below has been addressed in full generality by Itoh–Rouyer–Vîlcu [21]; see also [36, 38] and the discussion in [22, 31] concerning the curvature bounded below case. We prove Theorem 1.4 by giving an explicit formula for smoothing the vertices of a model polyhedral surface while respecting the CAT(κ) condition. Our approach also gives a similar conclusion for surfaces of curvature bounded below by κ, with the advantage of being more explicit than the approach in [21]; see Remark 7.2. Thus we provide a concrete and uniform approach to the general problem of smoothing vertices of polyhedra.

1.3. **The bigger picture.** We now elaborate more on the consequences of Theorem 1.1 and place our work within a larger context. While the CAT(κ) condition provides an effective axiomatic condition for studying metric surfaces, there is a variety of increasing general conditions that have also been studied in the literature. Indeed, the CAT(κ) condition is fairly restrictive; roughly speaking, a CAT(κ) surface is one that is the limit of spherical polyhedral surfaces of uniformly bounded area with vertices of negative curvature. We state the following omnibus theorem, which is a compilation of easy or known facts, along with Theorem 1.1. For simplicity, we assume that the metric surface X is closed.

Theorem 1.5. *Let X be a closed length surface and $\kappa \in \mathbb{R}$. The following conditions are such that each condition implies the next.*

(A) X is a smooth Riemannian 2-manifold with Gaussian curvature at most κ everywhere.
(B) X is a CAT(κ) surface.
(C) X is a surface of bounded curvature without cusp points.
(D) X is bi-Lipschitz equivalent to a constant curvature surface of the same topology. In particular, each point of X has a neighborhood bi-Lipschitz equivalent to a Euclidean disk.
(E) X is Ahlfors 2-regular and linearly locally contractible.
(F) X satisfies a quadratic isoperimetric inequality.
(G) X has locally finite Hausdorff 2-measure.

Let us define the terms in this theorem that are potentially less familiar or standardized. A metric space X is *Ahlfors 2-regular* if there is a constant $C > 0$ such that

$$\frac{1}{C} \cdot r^2 \leq |B(x, r)| \leq C \cdot r^2$$

for all $x \in X$ and $r \in (0, \operatorname{diam}(X))$. Here, $|A|$ denotes the Hausdorff 2-measure of a set A and $B(x, r)$ is the open ball at x of radius $r > 0$. The space X is *linearly locally contractible* if there exists $C > 0$ such that every ball $B(x, r)$ of radius $r \in (0, \operatorname{diam}(X)/C)$ is contractible in $B(x, Cr)$. We say that the space X *satisfies a quadratic isoperimetric inequality* if every point has a neighborhood V for which there is a constant $C > 0$ such that every closed Jordan curve Γ in V bounds a topological disk $U \subset V$ satisfying $|U| \leq C\ell(\Gamma)^2$.

The implication (A) $\Longrightarrow$ (B) is classical; see the appendix to Chapter II.1 of [10]. The implication (B) $\Longrightarrow$ (C) is a restatement of Theorem 1.1 together with the simple observation that a CAT(κ) surface does not have any so-called *cusp points*, i.e., points having total curvature 2π. Next, by a theorem of Reshetnyak [37, Theorem 9.10], the metric tangent space of every non-cusp point of a surface of bounded curvature is a Euclidean cone over a circle. That is, for each point x on the surface and all $\varepsilon > 0$ one can find $\delta > 0$ such that the ball $B(x, r)$ is $(1+\varepsilon)$-bi-Lipschitz equivalent to a neighborhood of the above Euclidean cone. See Lemma 6 in [12] for a proof of Reshetnyak's theorem. See also Bonk–Lang [7] for a strong quantitative version of this property. The implication (C) $\Longrightarrow$ (D) is shown by Burago in [12] using this local bi-Lipschitz equivalence. The implications (D) $\Longrightarrow$ (E) and (F) $\Longrightarrow$ (G) are elementary to verify from the definitions, while the implication (E) $\Longrightarrow$ (F) is due to Lytchak–Wenger [28, Corollary 5.5].

The final condition (G), namely, having locally finite Hausdorff 2-measure, represents the most general non-fractal metric surface. We briefly remark that fractal surfaces are a separate topic of current interest; these are generally more difficult to study and require separate methods. See for example the monograph of Bonk–Meyer [8]. Condition (G) is very general; nevertheless, the basic idea of the recent papers [16, 34, 33] is that a geometric study of surfaces analogous to that of surface of bounded curvature can be carried out using only the assumption of locally finite Hausdorff

2-measure. In fact, it turns out that the requirement of a length metric is not so essential and can be omitted.

Regarding condition (F), we mention a beautiful theorem of Lytchak–Wenger [27] that upper curvature bounds can be characterized in terms of isoperimetric inequalities. Namely, a proper length metric space is CAT(0) if and only if it satisfies the quadratic isoperimetric inequality (defined in a more general way) with the Euclidean constant $1/(4\pi)$. More generally, a proper length metric space is CAT(κ) if and only if its isoperimetric profile is no worse than that of the sphere of constant curvature κ.

One aspect of the theory concerns the topic of uniformization. The classical uniformization theorem states that any smooth Riemannian 2-manifold is conformally equivalent to a surface of constant curvature. A major theorem of Reshetnyak concerns conformal parametrizations of bounded curvature surfaces: for any surface of bounded curvature, its metric can be represented locally by a length element of the form $ds = e^{\lambda} ds_h$, where λ is the difference of two subharmonic functions and ds_h is the length element of a background smooth Riemannian metric. The global version of this theorem is due to Huber [20]. There are similar uniformization theorems for the more general classes of surfaces in Theorem 1.5, including the Bonk–Kleiner theorem for Ahlfors 2-regular, linearly locally contractible spheres [6]. A very general uniformization theorem applicable to any surface of locally finite Hausdorff 2-measure, yielding a so-called *weakly quasiconformal* parametrization from a constant curvature surface, has been proved in [34, 33].

We refer the reader to the following references for various non-trivial examples of surfaces illustrating the difference between the classes of surfaces in Theorem 1.5: Laakso [23] for a surface satisfying (E) but not (D); Theorem 1.1 in [15] for a surface satisfying (F) but not (E); Theorem 1.2 in [39] for a surface satisfying (G) but not (F). In addition, we observe that $\mathbb{R}^2$ equipped with any normed metric not arising from an inner product satisfies (D) but not (C). Moreover, any polyhedral surface with cone points of positive curvature satisfies (C) but not (B).

Finally, we remark that there have been several more advanced investigations of finite-dimensional CAT(κ) spaces in recent years; see especially [25, 26]. Two-dimensional metric spaces (not necessarily surfaces) satisfying the CAT(κ) condition have been studied in [32].

1.4. **Outline of the paper.** We give an overview of the organization of the paper. We collect preliminaries related to metric geometry in Section 2 and related to CAT(κ) spaces in Section 3. The material on CAT(κ) is probably well-known to experts but includes some propositions that we could not find in the literature. Section 4 describes our main tool of vertex-edge triangulations in preparation for the proof of Theorem 1.1.

The three proofs of Theorem 1.1 are given in Section 5. We describe briefly the main ideas of each proof. Let X be a CAT(κ) surface. The first proof uses two preliminary facts to obtain a non-sharp version of Theorem 1.2, namely Reshetnyak's majorization theorem and the Besicovitch inequality for metric spaces. The first of these implies that the excess of a triangle T in X is no larger than the area of the model triangle in the model space M^2_κ, while the second of these implies that area of T is not much small than the same model triangle provided that T has a uniform lower bound on its angles. The main task then is to partition an arbitrary triangle in finitely many subtriangles with uniformly large angles plus at most two small triangles of negligible excess. The second and third proofs rely on our notion of vertex-edge triangulations. While not every triangulation is a vertex-edge triangulation, we show that every triangulation has a vertex-edge refinement in a precise sense. The key property of these triangulations is that the sum of the areas of the collection of model triangles corresponding to a triangulation can only decrease under subdivision. On the other hand, the total excess of a triangulation can only increase under subdivision. This allows us to derive a uniform upper bound on the excess of a vertex-edge triangulation.

In the third proof, we prove that a CAT(κ) surface is the uniform limit of spherical polyhedral surfaces of locally uniformly bounded curvature, which is equivalent to being a surface of bounded curvature according to the standard definition. This third proof also relies on vertex-edge triangulations to control to total excess of the approximating surfaces.

The other results are proved in the following sections. First, we verify Proposition 1.3 on the equivalence of definitions of area in Section 6. Next, Theorem 1.4 on smooth approximation is proved in Section 7. Finally, we prove Theorem 1.2 relating the excess of a triangle to its area in Section 8.

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2. Background

2.1. **Basics of metric geometry.** We recall the basic definitions related to metric spaces. See standard references such as [10] and [11] for more detail. Let X be a set. A *metric* on X is a function $d\colon X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$

(1) $d(x, y) = 0$ if and only if $x = y$ (positive definite),
(2) $d(x, y) = d(y, x)$ (reflexive),
(3) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

The pair (X, d) is called a *metric space*. Most often, we refer to X itself as a metric space with the convention that d denotes the corresponding metric. We refer to $d(x, y)$ as the *distance* between x and y. Given $x \in X$ and $r > 0$, we let $B(x, r)$ denote the open metric ball centered at x of radius r. Note that this notation does not specify the metric space, though no confusion should result.

A *curve* in X is a continuous function from an interval to X. The *length* of a curve $\Gamma\colon [a, b] \to X$ is

$$\ell(\Gamma) = \sup \sum_{k=0}^{m} d(\Gamma(t_k), \Gamma(t_{k+1})),$$

where the supremum is taken over all finite subdivisions $a = t_0 < t_1 < \cdots < t_m = b$. The curve Γ is a *geodesic* if $\ell(\Gamma) = d(\Gamma(a), \Gamma(b))$. The notation $[xy]$ is used to denote a choice of geodesic from x to y, provided that such a geodesic exists. Note that in general such a geodesic need not be unique. In a slight abuse of notation, we also identify a geodesic $[xy]$ with its image.

The metric space X is a *length space* if

$$d(x, y) = \inf_{\Gamma} \ell(\Gamma)$$

for all $x, y \in X$, where the infimum is taken over all curves Γ connecting x and y. The space X is a *geodesic space* if every pair of points x, y can be joined by a geodesic $[xy]$. Observe in this case that $d(x, y) = \ell([xy])$.

Let (X, d), (Y, d') be metric spaces. A map $f\colon X \to Y$ is *L-Lipschitz* if

$$d'(f(p), f(q)) \leq L \cdot d(p, q)$$

for all $p, q \in X$. The map f is *L-bi-Lipschitz* if

$$\frac{1}{L} d(p, q) \leq d'(f(p), f(q)) \leq L d(p, q)$$

for all $p, q \in X$.

The diameter of a set $A \subset X$ is

$$\operatorname{diam} A = \sup_{x, y \in A} d(x, y).$$

The 2-*dimensional Hausdorff* δ-*content* of $A \subset X$ is

$$\mathcal{H}_\delta{}^2(A) = \inf \left\{ \sum_{j=1}^{\infty} c_2 \cdot \mathrm{diam}(A_j)^2 \right\},$$

where the infimum is taken over all countable collections of subsets $A_1, A_2, \ldots$ such that $A \subset \bigcup_{j=1}^{\infty} A_j$. The *Hausdorff* 2-*measure* of A is

$$\mathcal{H}^2(A) = \lim_{\delta \to 0} \mathcal{H}_\delta{}^2(A).$$

Here, $c_2 = \pi/4$ is a normalization constant so that the Hausdorff 2-measure on the plane coincides with 2-dimensional Lebesgue measure. Accordingly, we denote the Hausdorff 2-measure of a set A by $|A|$ rather than $\mathcal{H}^2(A)$ throughout this paper. The term "area" in this paper refers specifically to the Hausdorff 2-measure.

A useful fact about Hausdorff measure is the Besicovitch inequality. We state a two-dimensional version for metric spaces. See Theorem 2.1 in [34] for a proof.

Theorem 2.1 (Besicovitch inequality)**.** *Let X be a metric space homeomorphic to a closed disk with topological boundary ∂X. Suppose there is a closed Jordan domain $\Omega \subset \mathbb{R}^2$ and a L-Lipschitz bijective map from ∂X to $\partial \Omega$. Then*

$$|X| \geq \frac{4|\Omega|}{\pi L^2}.$$

2.2. **Polygons and triangulations.** Our focus in this paper is two-dimensional metric spaces. A *metric surface* is a metric space homeomorphic to a topological surface, or 2-manifold. Consistent with the main references such as [2], we assume that metric surfaces do not have boundary, although in principle the results of this paper should remain true for surfaces with boundary. We use the terms *length surface* and *geodesic surface* with the obvious meanings. For this section, we let X be a length surface.

Definition 2.2. Let $n \geq 2$ be an integer. A *(geodesic) n-gon* is a figure consisting of n points $v_1, \cdots, v_n$, called *vertices*, together with a collection of n geodesics joining them cyclically, called *edges*, contained in an ambient neighborhood $U \subset X$ homeomorphic to an open disk. In addition, we require that no edge may be crossed by the remaining edges, although two edges may intersect. An n-gon is also called a *polygon*. A polygon is *simple* if it forms a simple closed curve. A 3-gon with vertices x, y, z is called a *triangle* and denoted by $\triangle xyz$. A 4-gon is called a *quadrilateral*. The *perimeter* of a polygon is the sum of the lengths of its edges.

By the Jordan curve theorem, any simple polygon is the boundary of a closed Jordan domain P contained in the ambient neighborhood U. The interior of P is called the *interior* of the polygon, denoted by P°. Note that the previous fact depends on our assumption of an ambient disk neighborhood; otherwise, a simple polygon may not bound a unique disk, or any disk at all. For example, any three distinct point on a great circle will determine a triangle, but either hemisphere may be designated as the interior. In general, a non-simple polygon may not have a unique interior component, since the edges may overlap in a complicated way. However, we may define the interior of an arbitrary polygon as the set of points in the ambient disk U that the curve formed by the edges of the polygons winds around once.

We make the following convention: when we refer to a polygon as a set, we include the set of interior points in addition to the edge points. In a similar way, for a polygon P, we use ∂P to denote the set of edge points of P. Note that this does not coincide with the topological boundary of P if the polygon has "inward-pointing tails", although we do not encounter this situation in the present paper.

Two polygons are *non-overlapping* if they have disjoint interiors. A polygon $P \subset X$ is *convex* if any two points in P can be joined by a geodesic in P. This definition of convexity is sufficient for this paper, though we note that there are stronger notions of convexity that are essential for the study of general length surfaces. See Section 3 of [16] for more discussion.

We state a simple property of convexity.

Proposition 2.3. *Let P, S be two convex polygons in a metric surface with the property that any two points in $P \cup S$ can be joined by a unique geodesic. If $P^\circ \cap S^\circ$ is non-empty, then $P \cap S$ is a convex polygon.*

We omit the proof. Note that the conclusion may fail without the unique geodesic property.

Most of our proofs make use of covers of a surface by non-overlapping triangles, which we call *triangulations*. Our definition is somewhat different than the use of the term in general topology, since we do not require adjacent triangles to intersect along entire edges.

Definition 2.4. A *triangulation* of a subset $K \subset X$ is a locally finite collection of mutually non-overlapping convex triangles $\mathcal{T} = \{T_i\}$ such that $\bigcup_i T_i = K$. If the subset K itself is a polygon, then the vertices of K are called the *original vertices* of the triangulation $\mathcal{T}$. The *mesh* of $\mathcal{T}$ is $\sup_{T \in \mathcal{T}} \operatorname{diam} T$.

The union of the set of edges of a triangulation $\mathcal{T}$ form an embedded topological graph on the surface, called the *edge graph* of $\mathcal{T}$ and denoted by $\mathcal{E}$. The edge graph can be considered as a length space by giving it the length metric induced by the metric on X.

A general theorem on the existence of triangulations for arbitrary length surfaces was proved in [16]. We state a version that covers our purposes.

Theorem 2.5. *Let X be a length surface and $\mathcal{U}$ a cover of X by open sets. For all $\varepsilon > 0$ there exists a triangulation $\mathcal{T}$ of X with mesh at most ε and that each element $T \in \mathcal{T}$ is contained in a set $U \in \mathcal{U}$ satisfying $d(T, \partial U) \geq \varepsilon$. Moreover, if $\mathcal{E}$ is the edge graph of $\mathcal{T}$ with the induced length metric, then the inclusion of $\mathcal{E}$ into X is an ε-isometry (that is, every point in y is at distance at most ε from a point in $f(X)$).*

A map $f\colon X \to Y$ between metric spaces (X, d) and (Y, d') is a ε-isometry if $d(x,y) - \varepsilon \leq d'(f(x), f(y)) \leq d(x,y) + \varepsilon$ for all $x, y \in X$ and the image of f is ε-dense in Y.

The statement concerning the open cover $\mathcal{U}$ is not part of the triangulation theorem as stated in [16], although it follows from Remark 5.5 in [16]. Also, the triangulation theorem in [16] gives the stronger property of *boundary convexity* rather than just convexity, though this is not needed for our purposes. The final conclusion regarding the edge graph can be found in Proposition 5.2 in [34].

3. Preliminaries on CAT(κ) spaces

3.1. **Basic definitions.** In the section, we give a brief overview of CAT(κ) surfaces and collect various facts that are needed later. We refer the reader to Bridson–Haefliger [10] for further details.

For each $\kappa \in \mathbb{R}$, we let M^2_κ denote the 2-dimensional model space of constant curvature κ. More precisely, these are defined by letting M^2_0 be the Euclidean plane, M^2_1 be the 2-sphere, and M^2_{-1} be the hyperbolic plane. For $\kappa > 0$, $M^2_{\pm\kappa}$ is the rescaling of $M^2_{\pm 1}$ by the factor $\sqrt{\kappa}$. We let $\bar{d}_\kappa$ denote the metric on M^2_κ and D_κ denote the diameter of M^2_κ; that is, $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$ and $D_\kappa = \infty$ otherwise. It is often convenient to think of each model space as a conformal deformation of the Euclidean plane, that is, a smooth Riemannian metric obtained by rescaling pointwise the Euclidean metric on $\mathbb{R}^2$. Let ds_κ denote the corresponding length element for M^2_κ; then ds_κ is given by the formula

$$ds^2_\kappa = \frac{4(dx^2 + dy^2)}{(1 + \kappa(x^2 + y^2))^2}$$

defined for all $(x, y) \in \mathbb{R}^2$, with the additional requirement that $\sqrt{x^2 + y^2} < 1/\kappa$ if $\kappa < 0$. If $\kappa = 0$, this formula gives a rescaling of the usual Euclidean metric, but the resulting metric space is of course isometric to the Euclidean plane. If $\kappa > 0$, then we must add in the point ∞ to obtain the complete 2-sphere. One immediate consequence is that any two model spaces are locally bi-Lipschitz equivalent. More precisely, let $\kappa_1, \kappa_2 \in \mathbb{R}$, $x_1 \in M^2_{\kappa_1}$ and $x_2 \in M^2_{\kappa_2}$, and $0 < r < \min\{D_{\kappa_1}, D_{\kappa_2}\}$. Then the balls $B(x_1, r)$ and $B(x_2, r)$ are L-bi-Lipschitz equivalent, where $L \to 1$ uniformly (after fixing κ_1, κ_2) as $r \to 0$. In particular, for all $x_1 \in M^2_0$ and $x_2 \in M^2_1$, the ball $B(x_1, 2) \subset M^2_0$ is 2-bi-Lipschitz equivalent to the ball $B(x_2, \pi/2) \subset M^2_1$.

Next, let X be a metric space and fix $\kappa \in \mathbb{R}$. Given a triangle $T = \triangle pqr \subset X$ with perimeter at most $2D_\kappa$, a *comparison triangle in* M^2_κ for T is a triangle in M^2_κ with edges of matching lengths. We denote the comparison triangle of T by $\bar{T}_\kappa$. More precisely, let $\bar{p}, \bar{q}, \bar{r} \in M^2_\kappa$ be points satisfying $d(p, q) = \bar{d}_\kappa(\bar{p}, \bar{q})$, $d(p, r) = \bar{d}_\kappa(\bar{p}, \bar{r})$ and $d(q, r) = \bar{d}_\kappa(\bar{q}, \bar{r})$. Then we define $\bar{T}_\kappa = \triangle \bar{p}\bar{q}\bar{r}$. This triangle is unique up to an isometry of M^2_κ.

Given a point $x \in [pq]$, we let $\bar{x}$ denote the point on $[\bar{p}\bar{q}]$ such that $d(p, x) = \bar{d}_\kappa(\bar{p}, \bar{x})$ and $d(x, q) = \bar{d}_\kappa(\bar{x}, \bar{q})$. A triangle $T = \triangle pqr \subset X$ with perimeter at most $2D_\kappa$ is said to *satisfy the* CAT(κ) *inequality* if

$$d(x, y) \leq \bar{d}_\kappa(\bar{x}, \bar{y})$$

for all x, y in ∂T. A length metric space X is a CAT(κ) *space* if any two points in X at distance less than D_κ are joined by a geodesic and every triangle $T \subset X$ with perimeter less than $2D_\kappa$ satisfies the CAT(κ) inequality. We say that a length surface X is a *locally* CAT(κ) surface if every point $x \in X$ has a neighborhood U that is a CAT(κ) space as previously defined. For conciseness, we typically omit the word *locally* and refer to such a space a CAT(κ) *surface*.

3.2. **Angles.** Given a triangle $T = \triangle pqr \subset X$, the *model angle at* p, denoted $\bar{\angle}^\kappa_p(q, r)$, is the angle between the geodesics $[\bar{p}\bar{q}]$ and $[\bar{p}\bar{r}]$ in the comparison triangle $\triangle \bar{p}\bar{q}\bar{r}$ in M^2_κ. This is defined provided that the perimeter of T is at most $2D_\kappa$. In a CAT(κ) space, the model angle satisfies the triangle inequality: $\bar{\angle}^\kappa_p(q, r) \leq \bar{\angle}^\kappa_p(q, s) + \bar{\angle}^\kappa_p(s, r)$ for all p, q, r, s, provided that all these angles are defined.

For any metric space X, the *(Alexandrov) upper angle* between geodesics $[pq]$ and $[pr]$ is

$$\angle_p(q, r) = \limsup_{q', r' \to p} \bar{\angle}^0_p(q', r') \tag{3.1}$$

where $q' \in [pq] \setminus \{p\}$ and $r' \in [pr] \setminus \{p\}$. In general, the upper angle satisfies the triangle inequality: $\angle_p(q, r) \leq \angle_p(q, s) + \angle_p(s, r)$ for all points p, q, r, s such that the geodesics $[pq], [pr], [ps]$ exist. If the "lim sup" in (3.1) is an actual limit, then $\angle_p(q, r)$ is called the *angle* between $[pq]$ and $[pr]$. If X is a CAT(κ) space, the angle between adjacent sides of any geodesic triangle of perimeter less than $2D_\kappa$ always exists and is less than or equal to the corresponding model angle.

Let T be a triangle with interior angles α, β, γ (that is, the angles formed by each pair of consecutive edges). The *angle excess* or *excess* of T is $\delta(T) = \alpha + \beta + \gamma - \pi$. If $\mathcal{T}$ is a finite family of geodesic triangles, the *excess* of $\mathcal{T}$ is $\delta(\mathcal{T}) = \sum_i \delta(T_i)$. In connection with excess, we recall the classical Girard's theorem, which states that the area of a triangle in the 2-sphere M^2_1 is precisely equal to its excess. More generally, if T is a triangle in the model space M^2_κ for $\kappa > 0$, then $|T| = \delta(T)/\kappa$.

3.3. **Spherical law of cosines.** In this section, we restrict to the case where $\kappa > 0$. We recall the spherical law of cosines. See for example Section 1.1 of [3]. Consider a triangle with side lengths a, b, c and corresponding opposite angles α, β, γ in the model space M^2_κ. Then

$$\cos\gamma = \frac{\cos(c\sqrt{\kappa}) - \cos(a\sqrt{\kappa})\cos(b\sqrt{\kappa})}{\sin(a\sqrt{\kappa})\sin(b\sqrt{\kappa})}.$$

By expanding this equation with power series, we obtain the following inequality: for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $a, b < \delta$ then

$$(1 - \varepsilon)(a^2 + b^2 - 2ab\cos\gamma) \leq c^2 \leq (1 + \varepsilon)(a^2 + b^2 - 2ab\cos\gamma). \tag{3.2}$$

As a particular consequence, if $\gamma \geq \theta$ for some fixed value $\theta > 0$, then there is a corresponding constant $C_\theta > 0$ such that

$$C_\theta \min\{a^2, b^2\} \leq (1 - \varepsilon)(a^2 + b^2 - 2ab\cos(\gamma)) \leq c^2 \tag{3.3}$$

whenever $a, b < \delta$.

The following lemma forms the basis of the first proof of Theorem 1.1.

Lemma 3.1. *Fix $\kappa > 0$. For all $\theta > 0$, there exists $L \geq 1, \delta > 0$ such that the following holds for every triangle T in a $\mathrm{CAT}(\kappa)$ space with each angle at least θ and having perimeter at most δ.*

(1) The ratio of the length of any two edges of T is bounded by L.
(2) The canonical map from ∂T to its Euclidean model triangle $\partial \bar{T}_0$ is L-bi-Lipschitz.
(3) $16L^2|T| \geq 2|\bar{T}_0| \geq \delta(T)$.

The canonical map in (2) is the map that sends each edge of T to the corresponding edge of $\bar{T}_0$ preserving arc length, that is, each point $x \in \partial T$ to the point $\bar{x} \in \partial \bar{T}_0$.

Proof. By rescaling, we may assume that $\kappa = 1$. Choose $\delta > 0$ small enough so that (3.2) holds for $\varepsilon = 1/2$ for any triangle in M^2_κ with perimeter at most δ. Assume further that $\delta \leq 2D_\kappa$. Consider a triangle T with vertices p, q, r, with each angle at least θ and perimeter at most δ. Statement (1) follows immediately from (3.3).

We now verify statement (2). Let $\bar{T}_1$ be the spherical model triangle corresponding to T and let $\bar{T}_0$ be the Euclidean model triangle. Let $x, y \in \partial T$. If x, y belong to the same side, then $d(x, y) = \bar{d}_1(\bar{x}, \bar{y})$ and there is nothing more to prove. Otherwise, by symmetry, we may assume that the sides containing x and y, respectively, meet at a common vertex p. Let θ_0 denote the angle at the vertex $\bar{p}$ in $\bar{T}_1$. Let $\bar{T} = \triangle \bar{p}'\bar{x}'\bar{y}'$ denote the spherical model triangle corresponding to the triangle $\triangle pxy$, and $\bar{\theta}$ the angle at $\bar{p}'$. The $\mathrm{CAT}(\kappa)$ condition implies

$$d(x, y) = \bar{d}_1(\bar{x}', \bar{y}') \leq \bar{d}_1(\bar{x}, \bar{y})$$

and moreover that $\theta_0 \geq \bar{\theta} \geq \theta$. For the following inequalities, we use the auxiliary function

$$L(\rho) = d(p, x)^2 + d(p, y)^2 - 2d(p, x)d(p, y)\cos(\rho).$$

Applying (3.2) with the value $\varepsilon = 1/2$ gives

$$(1/2)L(\bar{\theta}) \leq \bar{d}_1(\bar{x}', \bar{y}')^2 = d(x, y)^2.$$

Since $\theta \leq \bar{\theta}$, the left-hand side of the above inequality is bounded below by $(1/2)L(\theta)$, which in turn has a lower bound of the form $C'_\theta(d(p, x)^2 + d(p, y)^2)$, C'_θ a constant depending only on θ. On the other hand, we have the upper bound $L(\theta_0) \leq d(p, x)^2 + d(p, y)^2$. By (3.2), we have $L(\theta_0) \geq (3/2)\bar{d}_1(\bar{x}, \bar{y})^2$. Combining these inequalities gives

$$\frac{L(\bar{\theta})}{2} = \frac{L(\bar{\theta})}{2L(\theta_0)} L(\theta_0) \geq \frac{C'_\theta}{2} \cdot \frac{2}{3} \bar{d}_1(\bar{x}, \bar{y})^2.$$

We conclude that $\bar{d}_1(\bar{x}, \bar{y}) \leq \sqrt{C'_\theta/3} \cdot d(x, y)$, hence that the canonical map from T to $\bar{T}_1$ is bi-Lipschitz with constant only depending on L. Then (2) follows since $\bar{T}_1$ and $\bar{T}_0$ are $\sqrt{2}$-bi-Lipschitz equivalent.

Finally, we show that (3) holds as a consequence of the Besicovitch inequality (Theorem 2.1). Observe that, since the perimeter of T is at most $\pi/2$, the triangles $\bar{T}_0$ and $\bar{T}_1$ are 2-bi-Lipschitz equivalent. Applying the Besicovitch inequality and Girard's theorem, we see that

$$|T| \geq \frac{|\bar{T}_0|}{4L^2} \geq \frac{|\bar{T}_1|}{16L^2} = \frac{\delta(\bar{T}_1)}{16L^2} \geq \frac{\delta(T)}{16L^2}.$$

This verifies (3). □

A similar statement, with more intricate proof, holds for surfaces of bounded curvature. See Lemma 4 in [5].

3.4. **Facts about** $\mathrm{CAT}(\kappa)$ **spaces.** We record a few foundational facts about $\mathrm{CAT}(\kappa)$ spaces. The first of these is Reshetnyak's majorization theorem; see Section 8.12 of [3].

Let $D \subset M_\kappa^2$ be a closed Jordan domain whose boundary has finite length, and let X be a metric space. A map $f\colon D \to X$ is *majorizing* if it is 1-Lipschitz and its restriction to the boundary ∂D is length-preserving. If Γ is a closed curve in X, we say that D *majorizes* Γ if there is a majorizing map $f\colon D \to X$ such that the restriction $f|_{\partial D}$ traces out Γ.

Theorem 3.2 (Reshetnyak's majorization theorem)**.** *For any closed curve Γ in a $\mathrm{CAT}(\kappa)$ space of length at most D_κ, there exists a convex region D in M_κ^2, and an associated map f such that D majorizes Γ under f.*

If Γ itself is the boundary of a simple triangle T, then it is easy to see that D is the model triangle $\bar{T}_\kappa$.

A consequence of Reshetnyak's majorization theorem is that any $\mathrm{CAT}(\kappa)$ surface has locally finite Hausdorff 2-measure. Consider a triangulation $\mathcal{T}$ of the surface as given by Theorem 2.5, where ε is chosen to satisfy $\varepsilon < D_\kappa/2$. For each triangle $T \in \mathcal{T}$, Reshetnyak's theorem gives a 1-Lipschitz map $\bar{T}_1 \to T$. Since $\bar{T}_1$ has finite area and $\mathcal{T}$ is locally finite, we see that the Hausdorff 2-measure on X is locally finite. Note however that we do not need the full strength of the triangulation theorem for this conclusion, but rather just the much easier property that any point lies in a polygonal neighborhood (Lemma 5.1 in [16]).

The next fact about $\mathrm{CAT}(\kappa)$ spaces concerns uniqueness of geodesics. If two points $x, y \in X$ satisfying $d(x,y) < D_\kappa$ are joined by a geodesic, then that geodesic is unique. The existence of a geodesic between sufficiently close points is a consequence of the Hopf–Rinow theorem; see Proposition 3.7 in [10]. Suppose that x, y belong to a neighborhood $U \subset X$ homeomorphic to a disk. If $d(x,y) \leq d(x, \partial U)$, then there is a geodesic in X from x to y. Moreover, the ball $B(x,r)$ is convex and contractible for all $r < \min\{d(x,\partial U)/2, D_\kappa/2\}$.

Lemma 3.3. *Let X be a $CAT(\kappa)$ space that is also a topological manifold. Then every geodesic can be extended indefinitely in both directions as a local geodesic. It is length-minimizing for all pairs of points at distance less than D_κ.*

See Proposition II.5.12 in [10] for a proof.

Corollary 3.4. *Let $T = \triangle pqr$ be a geodesic triangle and x an interior point. Then the geodesic $[px]$ can be extended to intersect the opposite side $[qr]$.*

Proof. By Lemma 3.3, $[px]$ can be extended indefinitely beyond x. Since the triangle T is compact, this extension must intersect the boundary ∂T at some point. This point must lies on the side $[yz]$ since otherwise the uniqueness of geodesic is violated. □

As a consequence of geodesic extendability, we can show that small triangles in $\mathrm{CAT}(\kappa)$ spaces are convex.

Lemma 3.5. *Let T be a triangle in a $\mathrm{CAT}(\kappa)$ space with perimeter at most $D_\kappa/2$. Assume further that T is contained in a neighborhood U homeomorphic to a closed disk with $d(T, \partial U) \geq \ell(\partial T)$. Then T is convex.*

Proof. Consider two points $x, y \in T$. Let Γ be the geodesic between them, which exists by the second assumption. We must show that the image of Γ is contained in T. If x, y belong to the same side of T, then Γ is just a subarc of this side and we are done. Assume then that x belongs to the side A and y belongs to the side B, with C the other side of T. Extend the side A in both directions until reaching ∂U. This divides U into two non-overlapping closed disks U_A^1 and U_A^2, where $T \subset U_A^1$. Do the same for B to obtain two non-overlapping closed disks U_B^1 and U_B^2. If Γ leaves the set U_A^1, then we homotope any subcurves of Γ in $(U_A^2)^\circ$ onto A to obtain a curve Γ_1 in U_A^1. It is clear that $\ell(\Gamma_1) \leq \ell(\Gamma)$. Next, if Γ_1 leaves the set U_B^1, we do the same procedure to obtain a curve Γ_2 in $U_B^1 \cap U_A^1$. This curve necessarily satisfies $\ell(\Gamma_2) \leq \ell(\Gamma_1)$. Finally, if the curve Γ_2 leaves T, it must be through the side C. We homotope any subcurve of Γ_2 in $U \setminus T$ onto C to obtain a curve Γ_3 contained in T. It is clear that $\ell(\Gamma_3) \leq \ell(\Gamma_2)$. □

3.5. **Surfaces of bounded curvature.** In this section, we give a brief overview of the theory of surfaces of bounded curvature. See the monographs [2] and [37] for a detailed treatment, as well as [40] for a shorter survey. The fundamental definition in the theory as developed by Alexandrov–Zalgaller is the following.

Definition 3.6. A length surface X has *bounded curvature* if for each $x \in X$ there is a neighborhood U containing x and a constant $C > 0$ such that $\delta(\mathcal{T}) \leq C$ for all finite collections $\mathcal{T}$ of mutually non-overlapping simple triangles contained in U.

The main method of proof used by Alexandrov–Zalgaller is polyhedral approximation. For a flat polyhedral surface Y (i.e., a polyhedral surface where each face is a Euclidean polygon), one can define the total curvature at a vertex to be 2π minus the total angle at that vertex. One can then define the curvature measure ω on Y as a signed Radon measure by defining $\omega(A)$ to be the sum of the total curvatures of all vertices in A. The measure ω in turn can be split into a positive part ω^+ and negative part ω^-, where ω^+ and ω^- are non-signed measures satisfying $\omega = \omega^+ - \omega^-$. Suppose that (X, d) is a length surface and (d_n) is a sequence of polyhedral metrics on X converging uniformly to d, with ω_n the corresponding curvature measures. We say that the sequence of metrics d_n has *locally uniformly bounded curvature* if for any compact set $A \subset X$ both $\omega_n^+(A)$ and $\omega_n^-(A)$ are uniformly bounded in n.

Surfaces of bounded curvature can be defined equivalently in terms of a sequence of polyhedral approximations. Namely, a length surface (X, d) has bounded curvature if and only if there is a sequence of polyhedral metrics d_n on X with locally uniformly bounded curvature that converges uniformly to d. The same statement is true with "polyhedral metric" replaced by "Riemannian metric"; in this case, the curvature measure is defined by integrating the Gaussian curvature with respect to area. Surfaces of bounded curvature can also be characterized in terms of the existence of conformal parametrizations due to Reshetnyak, which we discussed briefly in Section 1.3.

The characterization as limits of polyhedral surfaces enables one to define area and the curvature measure on a surface of bounded curvature by taking weak limits. We call the notion of area obtained this way the *Alexandrov area* and denote the area of a set A by $\mathrm{Area}(A)$. More precisely, let (d_n) be a sequence of polyhedral metrics on X with locally uniformly bounded curvature converging uniformly to a metric d, with φ_n denoting the transition map from (X, d_n) to (X, d) and ω_n the curvature measure for d_n. For all $A \subset X$ we define $\mathrm{Area}(A) = \lim_{n\to\infty} |\varphi_n^{-1}(A)|$. Likewise, we define $\omega(A) = \lim_{n\to\infty} \omega_n(\varphi_n^{-1}(A))$ It can be shown that these are independent of the choice of approximating surface and hence well-defined. In Section 6, we verify that the Alexandrov area coincides with the Hausdorff 2-measure for all $\mathrm{CAT}(\kappa)$ surfaces.

We mention a few more advanced concepts related to surfaces of bounded curvature. First is the notion of sector angle. Let $p \in X$ be a point, let $[pq]$ and $[pr]$ be geodesics intersecting only at p, and let O be a closed oriented Jordan neighborhood of p divided into two sectors U and V. One of these, say U, contains the arc of ∂O running counterclockwise from $[pq]$ to $[pr]$. We define the sector angle of $[pq]$ and $[pr]$ as follows. Let $a_0, \ldots, a_n$ be a sequence of points in $\partial U \cap \partial O$ in cyclic order with $a_0 \in [pq]$ and $a_n \in [pr]$, and $[pa_1], \ldots, [pa_n]$ be a sequence of geodesics intersecting ∂U only at their endpoints and intersecting each other only at p. The *sector angle* of $[pq]$ and $[pr]$, denoted by $\widetilde{\angle}_p(q, r)$, is the supremum of the sum $\sum_{i=1}^n \angle_p(a_i, a_{i+1})$, where the supremum is taken over all such n-tuples $a_0, \ldots, a_n$ and admissible choices of geodesics. The sector angle is independent of the choice of neighborhood O.

Next, the *total angle at a point* p is the supremum of the sum of sector angles taken over all partitions of a neighborhood of p into sectors. We record the observation that if p is the interior point of a geodesic, then the sector angle at p is at least 2π, since the sector corresponding to each side of the geodesic contributes at least π to the total angle.

Let X be a surface of bounded curvature and let D be a polygon. Denote the sector angle at a vertex v_i of ∂D on the interior side of ∂D by ϕ_i. Assume further that the interior sector angle of each non-vertex point of ∂D is π. We define the *interior rotation* of the curve ∂D by $\tau(\partial D) = \sum_i (\pi - \phi_i)$. The interior rotation can be defined for more general simple closed curves by taking limits of polygons, though we do not need this generality here.

The following version of the Gauss–Bonnet theorem holds for surfaces of bounded curvature.

Theorem 3.7 (Gauss–Bonnet)**.** *If X is a surface of bounded curvature and $D \subset X$ is a polygon, then*

$$\tau(\partial D) + \omega(D^\circ) = 2\pi.$$

4. Vertex-Edge Triangulations

In this section, we define vertex-edge triangulations and establish their main properties in preparation for the second and the third proofs of Theorem 1.1. First, we formally state the relevant definitions. Throughout this section, X is a CAT(κ) surface.

Definition 4.1. Let $P \subset X$ be a convex polygon. A *vertex-edge partition* of P is a cover of P by non-overlapping convex polygons that can be obtained by the following inductive process. First, $\mathcal{T}_0 = \{P\}$ itself is a vertex-edge partition.

> *Vertex-edge subdivision.* Let $\mathcal{T}$ be a vertex-edge partition. Given a polygon $T \in \mathcal{T}$, we subdivide it by connecting one of its vertices to a point on an edge not containing that vertex by a geodesic in T that intersects ∂T only at the endpoints. This produces two non-overlapping convex polygons T_1 and T_2 whose union is T. We define a new vertex-edge partition $\mathcal{T}'$ by replacing T with the two polygons T_1, T_2.

If a vertex-edge partition $\mathcal{T}$ of P consists only of triangles, then we call it a *vertex-edge triangulation* of P.

We make the following useful observation: if P is a triangle, then every intermediate polygon appearing in the vertex-edge subdivision is also a triangle.

Definition 4.2. Suppose that $\mathcal{A}$ is a family of non-overlapping convex polygons contained in a polygon P. A *vertex-edge refinement* of $\mathcal{A}$ with respect to P is a triangulation $\mathcal{T}$ with the following properties:

(1) $\mathcal{T}$ is a vertex-edge triangulation of P;
(2) Each polygon $A \in \mathcal{A}$ is the union of triangles in $\mathcal{T}$.
(3) For each $A \in \mathcal{A}$, the set $\mathcal{T}_A = \{T \in \mathcal{T} : T \subset A\}$ is itself a vertex-edge triangulation of A.

The point of this definition is that $\mathcal{T}$ is simultaneously a vertex-edge triangulation of the original triangle T and each of the polygons in $\mathcal{A}$. In most subsequent proofs $\mathcal{A}$ itself will be a triangulation of P (not necessarily vertex-edge).

4.1. **Existence of vertex-edge refinements.**

First, we have a lemma about recognizing vertex-edge triangulations.

Lemma 4.3. *Let P be a polygon in a* CAT(κ) *surface of perimeter less than $2D_\kappa$ such that P is the union of non-overlapping convex polygons $P_1, \ldots, P_n$, where each P_i is a triangle or quadrilateral. Assume that each polygon P_i intersects P_{i+1} along a single common edge, and that the P_i do not intersect otherwise. Moreover, if P_i is a quadrilateral, then we assume that it does not intersect P_{i-1} and P_{i+1} along adjacent edges. Finally, we require each original vertex of P to be a vertex of one of the P_i. We form a triangulation $\mathcal{T}$ of P by dividing each P_i that is a quadrilateral into triangles by connecting two non-adjacent vertices with a geodesic. Then $\mathcal{T}$ is a vertex-edge triangulation of P.*

Proof. Observe first that the perimeter assumption on P implies that P has at least three original vertices.

The lemma follows by induction on n, the number of subpolygons comprising P. It is clear that the conclusion holds if $n = 1$. Assume next that the conclusion holds whenever $n \leq N - 1$ for some integer $N \geq 2$. Let P be a polygon comprised of $n = N$ subpolygons as in the statement of the lemma. In the first case, suppose that P has an original vertex that is a vertex of a polygon P_i for some $i \in \{2, \ldots, n-1\}$. This vertex must be contained in a common edge with P_{i-1} or P_{i-1}. Subdivide P along this common edge, giving two subpolygons handled by the inductive hypothesis.

Otherwise, either P_1 or P_n is a quadrilateral containing two original vertices of P not in common with P_2 or P_{n-1}, respectively. Say that P_1 has this property. Let e be the edge of $\mathcal{T}$ crossing through P; then e must connect one original vertex of P and one vertex of the common edge with P_2. So we subdivide P first along e, then along the common edge with P_2. This divides P into the two subtriangles of P_1 and the polygon $P' = P_2 \cup \cdots \cup P_n$, which is handled by the inductive hypothesis. □

The main result of this section is the following.

Theorem 4.4. *If T is a convex triangle of perimeter less than $2D_\kappa$ that is a $\mathrm{CAT}(\kappa)$ space, then any finite family $\mathcal{A}$ of non-overlapping convex polygons contained in T has a vertex-edge refinement with respect to T.*

The proof is very technical, but the geometric idea at its core can be very easily read off from Figure 4.1

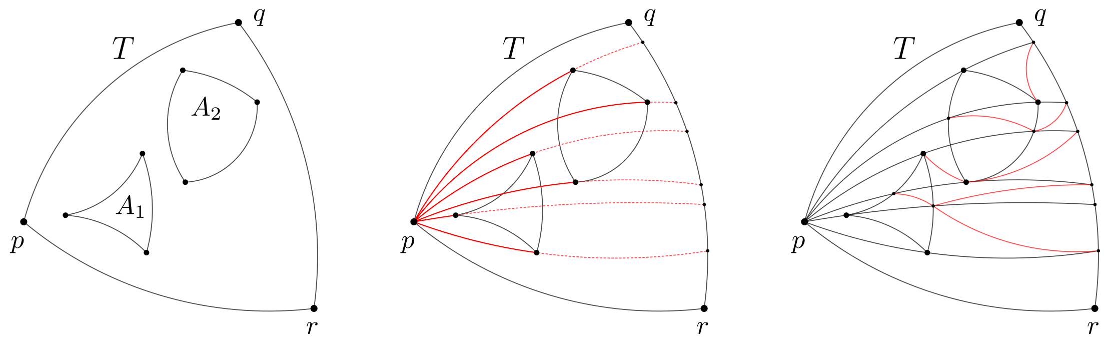
Proof. Let $T = \triangle pqr$ be such a triangle and let $\mathcal{A} = \{A_i\}_{i=1}^n$ be a finite collection of non-overlapping convex polygons in T. Enumerate the vertices of the A_i as $\{v_j\}_{j=1}^m$. We give the following procedure to find a vertex-edge refinement. Note that our assumption on T implies that any pair of points in T are connected by a unique geodesic.

(1) For each vertex v_j not contained in $[pq]$ or $[pr]$, extend the unique geodesic $[pv_j]$ to a geodesic that intersects the opposite edge $[qr]$ using Lemma 3.3. See Figure 4.1b. This yields a collection of distinct geodesics $\{I_k\}_{k=1}^K$, enumerated in increasing order based upon where the endpoint lies on $[qr]$, oriented from q to r. Note that two geodesics I_{j_1}, I_{j_2} cannot share the same endpoint, since otherwise they must coincide by uniqueness of geodesics. Also, set $I_0 = [pq]$ and $I_{K+1} = [pr]$. For all $k \in \{0, \ldots, K\}$, the geodesics I_k, I_{k+1} together with a subarc of $[qr]$ bound a triangle T_k.
(2) For each triangle T_k apply the following. Let $\mathcal{A}_k$ denote the collection of polygons of the form $A_i \cap T_k$ intersecting the interior of T_k; denote this polygon by T_k^i. By Proposition 2.3, T_k^i is a convex polygon. Note that each vertex of each T_k^i is contained in $I_k \cup I_{k+1}$. This implies that every edge of some T_k^i not contained in I_k or I_{k+1} must connect I_k to I_{k+1}. Enumerate such distinct edges as $\{J_k^j\}_{j=1}^{J_k}$ in increasing order from $[qr]$ to the vertex p. Also, denote the subarc of $[qr]$ contained in T_k by J_k^0. Note that J_k^j and J_k^{j+1} along with the relevant subarcs of I_k and I_{k+1} form either a triangle or quadrilateral. We subdivide each triangle T_k as follows. For all $j \in \{0, \ldots, J_k - 1\}$, if J_k^j and J_k^{j+1} do not share an endpoint, then connect an endpoint of J_0 to an endpoint of J_1 by a geodesic, forming two triangles. In this manner we obtain a triangulation $\mathcal{T}$ of P. The final product is shown in Figure 4.1c.

It follows from construction that $\mathcal{T}$ is a vertex-edge triangulation of T. Indeed, $\mathcal{T}$ is obtained first by drawing geodesics from p to the opposite edge to form the subtriangles T_k. Then each T_k is the union of triangles satisfying the requirements of Lemma 4.3. We conclude that $\mathcal{T}$ is vertex-edge.

The final property is that, for each $A \in \mathcal{A}$, the collection $\mathcal{T}_A$ can be obtained by vertex-edge subdivision of A. Observe that the geodesics $\{I_k\}_{k=1}^K$ divide A into the non-overlapping union of triangles and quadrilaterals satisfying the requirements of Lemma 4.3. We now apply this lemma to conclude that $\mathcal{T}_A$ is a vertex-edge triangulation of A. □

4.2. **Subdividing triangles.** The main usefulness of vertex-edge triangulations stems from the following lemma, which states that both excess and model area behave monotonically when subdividing triangles.



(a) Initial setup (b) Drawing the initial geodesics (c) The final triangulation

Figure 4.1

Lemma 4.5. *Let $T = \Delta pqr$ be a triangle of perimeter at most $2D_\kappa$ satisfying the $\mathrm{CAT}(\kappa)$ condition and s be a point on the side $[qr]$. Let $\mathcal{T}$ be the vertex-edge triangulation consisting of the triangles $T_1 = \Delta psq$ and $T_2 = \Delta psr$ formed by connecting p and s. then*

(1) Total excess does not decrease:

$$\delta(T) \leq \delta(T_1) + \delta(T_2)$$

(2) Total model area does not increase:

$$|\bar{T}_1| + |\bar{T}_2| \leq |\bar{T}|,$$

where $\bar{T}, \bar{T}_1, \bar{T}_2$ are the corresponding comparison triangles in the model space M^2_κ.

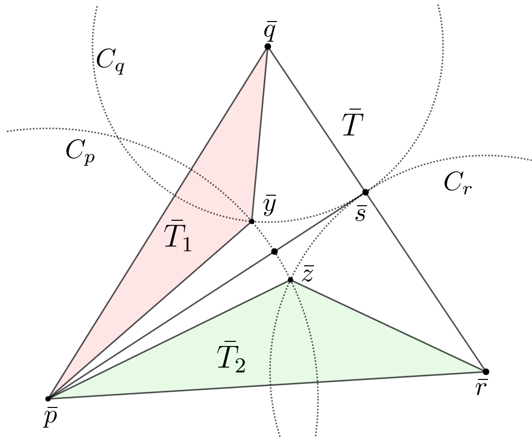
Proof. We first prove (1). The triangle inequality for angles implies that $\alpha \leqslant \alpha_1 + \alpha_2$ and $\pi \leqslant \gamma_1 + \beta_2$. It follows that

$$\begin{aligned}
\delta(T) &= \alpha + \beta + \gamma - \pi \\
&\leqslant \alpha_1 + \alpha_2 + \beta + \gamma + (\gamma_1 + \gamma_2 - \pi) - \pi \\
&= (\alpha_1 + \gamma_1 + \beta - \pi) + (\alpha_2 + \beta_2 + \gamma - \pi) \\
&= \delta(T_1) + \delta(T_2).
\end{aligned}$$

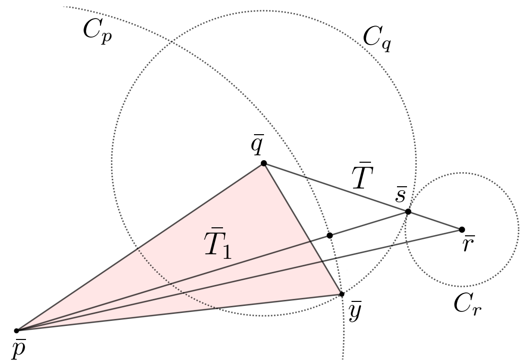
Next, to prove (2), the main task is to show that the comparison triangles $\bar{T}_1$ and $\bar{T}_2$ can be fitted into $\bar{T}$ such that $\bar{T}_1$ and $\bar{T}_2$ are mutually non-overlapping. The situation is summarized in Figure 4.2a.

Denote by $C_{\bar{p}}, C_{\bar{q}}, C_{\bar{r}}$ the circles centered at the comparison points $\bar{p}, \bar{q}, \bar{r}$, respectively, of radius $d(p,s)$, $d(q,s)$, $d(r,s)$, respectively. Observe that $C_{\bar{q}}$ and $C_{\bar{r}}$ intersect at a point $\bar{s}$ lying on $[\bar{q}\bar{r}]$. Next, the triangle inequality and the $\mathrm{CAT}(\kappa)$ condition implies that the circles $C_{\bar{p}}$ and $C_{\bar{q}}$ intersect at a point $\bar{y}$ on the arc on $C_{\bar{q}}$ from $\bar{s}$ to $[\bar{p}\bar{q}]$ that intersects the interior of $\bar{T}$. Likewise, the circles $C_{\bar{p}}$ and $C_{\bar{r}}$ intersect at a point $\bar{z}$ on the arc of $C_{\bar{r}}$ from $\bar{s}$ to $[\bar{p}\bar{r}]$ that intersects the interior of $\bar{T}$. We identify the model triangle $\bar{T}_1$ with the triangle $\triangle\bar{p}\bar{q}\bar{y}$ and the model triangle $\bar{T}_2$ with the triangle $\triangle\bar{p}\bar{r}\bar{z}$.

We claim that $\bar{T}_1$ and $\bar{T}_2$ are non-overlapping. To see this, consider the geodesic perpendicular to $[\bar{q}\bar{r}]$ passing through $\bar{s}$. This geodesic splits $\bar{T}$ into two non-overlapping polygons. The points $\bar{y}$ and $\bar{q}$, lie in one polygon, while $\bar{z}$ and $\bar{r}$ lie in the other. We see from this that the geodesics $[\bar{p}\bar{q}]$, $[\bar{p}\bar{y}]$, $[\bar{p}\bar{z}]$, $[\bar{p}\bar{r}]$ emanate from $\bar{p}$ in cyclic order. Since $\bar{T}_1$ is contained in the sector determined by $[\bar{p}\bar{q}]$ and $[\bar{p}\bar{y}]$, and $\bar{T}_2$ is contained in the sector determined by $[\bar{p}\bar{z}]$ and $[\bar{p}\bar{r}]$, we conclude that $\bar{T}_1$ and $\bar{T}_2$ do not overlap. In particular, a configuration like that in Figure 4.2b is not possible.



(a) Subdividing a CAT(κ) triangle



(b) An impossible configuration

Figure 4.2

Since $\bar{T}_1$ and $\bar{T}_2$ are disjoint, we see in particular that the segment $[\bar{p}\bar{r}]$ does not intersect $\bar{T}_1$. Thus the triangle $\bar{T}_1$ is contained in $\bar{T}$. Likewise, the triangle $\bar{T}_2$ is contained in $\bar{T}$. Since $\bar{T}_1$ and $\bar{T}_2$ are non-overlapping, it follows that $|\bar{T}_1| + |\bar{T}_2| \leq |\bar{T}|$. □

As an immediate consequence of the previous lemma, we have that if $\mathcal{T}$ is a vertex-edge triangulation of the triangle T, then

$$\sum_{T_i \in \mathcal{T}} \delta(T_i) \leqslant \kappa \cdot \text{Area}(\bar{T}).$$

5. CAT(κ) surfaces have bounded curvature

5.1. **The first proof.** Let (X, d) be a CAT(κ) surface. Without loss of generality, we assume that $\kappa = 1$. Given a point $x \in X$, let U be a neighborhood of x with compact closure. For such a neighborhood $|U|$ is finite. We may choose U to be sufficiently small so that $\delta(T) \leq \pi/20$ for any triangle $T \subset U$, Lemma 3.1 holds for $\theta = \pi/20$, and the model triangles $\bar{T}_0$ and $\bar{T}_1$ are $\sqrt{2}$-bi-Lipschitz equivalent for any triangle $T \subset U$.

To verify that X has bounded curvature, it is enough to show that there is a constant $C \geq 1$ such that

$$\delta(T) \leq C|T| \tag{5.1}$$

for all simple triangles $T \subset U$. Let T be such a triangle. If $\delta(T) \leq 0$, then the claim holds trivially. Thus we may assume that $\delta(T) > 0$. Ideally, we would be able to obtain (5.1) as an consequence of Lemma 3.1. The obstacle is that T may have a small angle. The strategy of our proof is to subdivide T into two parts: a collection of triangles to which Lemma 3.1 applies and at most two triangles of small diameter.

Denote the angles of T by α, β, γ, arranged so that $\alpha \geq \beta \geq \gamma$. Let p, q, r denote the vertices at α, β, γ, respectively, and A, B, C the edges opposite α, β, γ, respectively. Observe in particular that $\alpha \geq \pi/3$. In the case that $\gamma \geq \pi/20$, we apply Lemma 3.1 (3) to conclude that $|T| \geq \delta(T)/(8L^2)$ for some fixed constant L. This verifies (5.1) in the special case that $\gamma \geq \pi/20$.

Thus we assume in the following that $\gamma \leq \pi/20$. As a consequence, observe that $\alpha \in [9\pi/20, \pi)$ (the possibility $\alpha = \pi$ violates the uniqueness of geodesics for CAT(κ) surfaces). In the case that $\alpha \geq 14\pi/20$, we perform an initial subdivision of T. Let E be an arc that bisects the angle α and intersects A at a point e. This forms two subtriangles $T_1 = [pqe]$ and $T_2 = [pre]$. Let α_1 denote the angle at p in the triangle T_1 and α_2 the same for T_2. Then $\alpha_1, \alpha_2 \in [7\pi/20, \pi/2)$. Note that

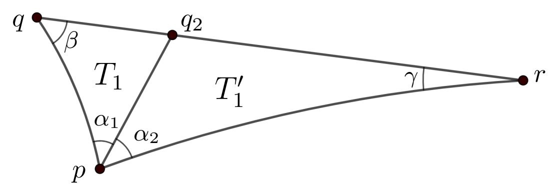


Figure 5.1. The triangle T in the first proof, following reductions

$\beta \leq 6\pi/20$. We see that for T_1, one of the three possibilities holds: the excess $\delta(T_1)$ is negative; each angle of T_1 is at least $\pi/20$; or, the angle β is at most $\pi/20$. The first two cases are handled by the previous observations. In turn, for the triangle T_2, we already have that $\gamma \leq \pi/20$. Since $\delta(T) \leq \delta(T_1) + \delta(T_2)$, it is enough to prove (5.1) for the triangles T_1 and T_2 separately.

In this way, we reduce to the case that T has largest angle $\alpha \in [9\pi/20, 14\pi/20]$ and smallest angle $\gamma \leq \pi/20$. Observe as a consequence that $\beta \in [7\pi/20, 11\pi/20]$. See Figure 5.1.

We apply the following procedure to split off good triangles from T. Assume we have a triangle $T = T_0'$ such that $\alpha, \beta \in [4\pi/20, 17\pi/20]$. By relabeling vertices if needed, we may assume that $\alpha \geq \beta \geq \gamma$, and in particular (given that $\delta(T) > 0$) that $\alpha \geq 9\pi/20$. Let C_1 be the arc bisecting the angle α, which intersects A at a point q_2. This splits T into two triangles $T_1 = [pq_2q]$ and $T_1' = [pq_2r]$. Let β_1 denote the angle at q_2 in T_1 and β_1' denote the angle at q_2 in T_1'. Define α_1, α_1' similarly. Note that $\alpha_1' \geq 4\pi/20$, which implies that $\beta_1' \leq 17\pi/20$. This in turn implies that $\beta_1 \geq 3\pi/20$. Thus all the angles of T_1 are at least $3\pi/20$, and we can apply Lemma 3.1 to T_1. On the other hand, the triangle T_1' satisfies $\alpha_1', \beta_1' \in [4\pi/20, 17\pi/20]$, the same angle conditions as T itself. Let p_1 denote the vertex of T_1 lying on the edge B and q_1 denote the vertex on the edge A. Thus we may repeat the procedure to obtain triangles $T_2, T_3, T_4, T_5, \ldots$, vertices $p_2, q_2, p_3, q_3, \ldots$ and so forth. By Lemma 3.1, the lengths of each of the edges T_n are comparable.

We need to show that we can terminate this process after finitely many steps. More precisely, we claim that for all $\varepsilon > 0$ one can find $n \in \mathbb{N}$ such that $\delta(T_n) < \varepsilon$, which follows if we can show that $\operatorname{diam}(T_n') \to 0$ as $n \to \infty$. First, by the comparability of the edge lengths of T_n, we can rule out the case that $p_n \to r$ but q_n does not, or vice versa. Next, the same comparability rules out the case that both p_n and q_n converge to a limit point other than r. We conclude that both p_n and q_n converge to r. This implies that $\operatorname{diam} T_n' \to 0$.

5.2. **The second proof.** Let X be a CAT(κ) surface and let $x \in X$. Let $r > 0$ be sufficiently small so that $B(x, r)$ is CAT(κ) and convex in X. Because the proof only uses the positivity of κ, to simplify notation we may assume without loss of generality that $\kappa = 1$. Use Theorem 2.5 to find a triangulation $\mathcal{S}$ of $B(x, r)$ with mesh less than $2D_\kappa$. Let $S_1, \ldots, S_n$ be the collection of triangles in $\mathcal{S}$ that contain x. Then $S = \bigcup_{i=1}^n S_i$ is a closed topological disk such that $x \in S^\circ$. This follows from the convexity of each triangle and uniqueness of geodesics. We claim that S° is the neighborhood of x as required by the definition of surface of bounded curvature.

Let $\mathcal{T} = \{T_j\}_{j=1}^m$ be an arbitrary collection of simple non-overlapping triangles in S°. Our goal is to find a uniform upper bound on $\delta(\mathcal{T})$. We apply Theorem 4.4 to find for each $T_j \in \mathcal{T}$ a triangulation $\mathcal{T}_j$ of T_j that is a vertex-edge refinement of the collection of polygons $S_i \cap T_j$ intersecting the interior of T_j with respect to T_j. Note that Proposition 2.3 implies that each set $S_i \cap T_j$ is in fact a polygon provided that its interior is non-empty. Observe from Lemma 4.5 that $\delta(T_j) \leq \delta(\mathcal{T}_j)$.

Next, for each triangle S_i, let $\mathcal{S}_i$ be the set of triangles in $\bigcup_j \mathcal{T}_j$ contained in S_i. We apply Theorem 4.4 a second time to $\mathcal{S}_i$ to find a vertex-edge refinement of $\mathcal{S}_i$ with respect to S_i. Denote this by $\mathcal{S}_i'$. Suppose that $T \in \mathcal{T}_j$ and contained in S_i. Let $\mathcal{S}_i'(T)$ denote the subset of $\mathcal{S}_i'$ contained in T. Then, according to Theorem 4.4, $\mathcal{S}_i'(T)$ is a vertex-edge triangulation of T and hence $\delta(T) \leq \delta(\mathcal{S}_i'(T))$

by Lemma 4.5. This gives

$$\sum_{j=1}^{m} \delta(T_j) \le \sum_{j=1}^{m} \sum_{T\in\mathcal{T}_j} \delta(T) \le \sum_{i=1}^{n} \sum_{T\in\mathcal{S}_i} \sum_{T'\in\mathcal{S}'_i(T)} \delta(T') \le \sum_{i=1}^{n} \sum_{T\in\mathcal{S}_i} \sum_{T'\in\mathcal{S}'_i(T)} |\bar{T}'_1|.$$

Here, $\bar{T}'_1$ is the model triangle in M^2_1 corresponding to T'. Similarly, for each triangle S_i we let $\bar{S}_i$ denote the model triangle of S_i in M^2_1. Since $\mathcal{S}'_i$ is a vertex-edge refinement of S_i, Lemma 4.5 gives

$$\sum_{i=1}^{n} \sum_{T\in\mathcal{S}_i} \sum_{T'\in\mathcal{S}'_i(T)} |\bar{T}'_1| \le \sum_{i=1}^{n} \sum_{T'\in\mathcal{S}'_i} |\bar{T}'_1| \le \sum_{i=1}^{n} |\bar{S}_i|.$$

Combining these inequalities gives an upper bound for $\delta(\mathcal{T}) = \sum_{j=1}^{m} \delta(T_j)$ independent of the choice of $\mathcal{T}$. This completes the proof.

5.3. **The third proof.** As noted above, this proof is partly based on discussions with François Fillastre [19] and a MathOverflow post by Anton Petrunin [35].

Let X be a locally CAT(κ) surface. Let $\mathcal{U}$ be a cover of X by convex open sets of diameter less than $2D_\kappa$ on which the restriction of the metric is CAT(κ). Let ε_n be a sequence of decreasing reals each less than $2D_\kappa$ and limiting to 0. Apply Theorem 2.5 to obtain a sequence of triangulations $\mathcal{T}_n$ of X corresponding to the parameters ε_n and the collection $\mathcal{U}$; in particular, each $\mathcal{T}_n$ has mesh less than $2D_\kappa$. By Theorem 4.4, we can choose the $\mathcal{T}_n$ so that each triangle $S \in \mathcal{T}_n$ for all $n \ge 2$ is contained in some triangle of $\mathcal{T}_1$, and so that for each $T \in \mathcal{T}_1$ the subset of $\mathcal{T}_n$ of triangles intersecting the interior of T form a vertex-edge triangulation of T.

Let $\mathcal{E}_n$ be the edge graph of $\mathcal{T}_n$. According to Theorem 2.5, the induced length metric $\widetilde{d}_n$ on $\mathcal{E}_n$ satisfies

$$d(x,y) \le \widetilde{d}_n(x,y) \le d(x,y) + \varepsilon_n$$

for all $x, y \in \mathcal{E}$. Define the surface X_n by, for each $T \in \mathcal{T}_n$, gluing in a model triangle of constant curvature κ having the same edge lengths into the edge graph $\mathcal{E}_n$. Identify $\mathcal{E}_n$ with a subset of X_n in the natural way, and X_n with X by choosing some bijection that is the identity map on the edge graph $\mathcal{E}_n$. Having made these identifications, denote the new metric on X by $\bar{d}_n$.

It follows from the CAT(κ) condition that $d(x,y) \le \bar{d}_n(x,y)$ for all $x, y \in \partial T$ for some triangle $T \in \mathcal{T}_n$. Given two points $x, y \in X$, let $\bar{\Gamma}_n$ be a curve from x to y whose $\bar{d}$-length satisfies $\ell_{\bar{d}_n}(\Gamma) \le \bar{d}_n(x,y) + \varepsilon_n$. We assume that the restriction of $\bar{\Gamma}_n$ to each triangle $T \in \mathcal{T}_n$ is a $\bar{d}_n$-geodesic. Define the curve Γ by replacing the subcurve of $\bar{\Gamma}_n$ intersecting a given triangle T with the d-geodesic connecting the same endpoints. Then Γ satisfies $\ell_d(\Gamma) \le \ell_{\bar{d}_n}(\bar{\Gamma}_n) + 2\varepsilon_n$. Combining these inequalities gives

$$d(x,y) \le \ell_d(\Gamma) \le \ell_{\bar{d}_n}(\bar{\Gamma}_n) + 2\varepsilon_n \le \bar{d}_n(x,y) + 3\varepsilon_n.$$

Next, for any points $x, y \in X$, we can find points $x', y' \in \mathcal{E}_n$ such that $\bar{d}_n(x,x') \le \varepsilon_n$ and $\bar{d}_n(y,y') \le \varepsilon_n$. So for all $x, y \in X$, we have

$$\bar{d}_n(x,y) \le \bar{d}_n(x',y') + 2\varepsilon_n \le \widetilde{d}_n(x',y') + 2\varepsilon_n \le d(x',y') + 3\varepsilon_n \le d(x,y) + 5\varepsilon_n.$$

We conclude that the spaces X_n converge uniformly to X.

Finally, we claim that the surfaces X_n have locally uniformly bounded curvature. Let $x \in X$. Let V be the interior of the union of all triangles in $\mathcal{T}_1$ containing x, observing that $x \in V$ and that its closure $\overline{V}$ is a closed polygon containing x. For all $n \in \mathbb{N}$, let V_n denote the set V considered with the metric $\bar{d}_n$. Note that each vertex of X_n has total angle at least 2π and hence negative curvature. So all the positive curvature on X_n comes from the interior of the faces. Since $\mathcal{T}_n$ is a vertex-edge triangulation, we have the uniform bound $|V_n| \le |V_1|$. We conclude the positive part of the curvature on V_n is at most $\kappa|V_1|$. In summary, X is the uniform limit of surfaces X_n of locally uniformly bounded curvature, and hence X itself has bounded curvature.

6. Hausdorff area vs. Alexandrov area

In this section, we verify that the Hausdorff area of a surface X of bounded curvature without cusp points coincides with the Alexandrov area. This follows easily from Reshetnyak's theorem that the metric tangent at each point is a cone over a circle. Recall that, for a set $A \subset X$, we use $|A|$ to denote the Hausdorff area (i.e., 2-measure) and $\text{Area}(A)$ to denote Alexandrov area. We must show that $|A| = \text{Area}(A)$ for an arbitrary set. Since both Hausdorff area and Alexandrov area are determined by their values on polygons, without loss of generality we may assume that A is a polygon.

Let $A \subset X$ be a polygon. We can consider A as its own metric space with the induced length metric; this does not affect either notion of area. Let $\varepsilon > 0$ be given. For each $x \in X$, let B_x be an open metric ball at x with the property of being $(1+\varepsilon)$-bi-Lipschitz equivalent to a ball in a Euclidean cone over a circle, and let $\mathcal{U} = \{B_x : x \in A\}$. Let $\mathcal{T}$ be a triangulation of A subordinate to the cover $\mathcal{U}$ and having mesh ε. Define a polyhedral surface A_ε by replacing each triangle $T \in \mathcal{T}$ with the corresponding model triangle $\bar{T}_0$. It is clear that $|A_\varepsilon| = \text{Area}(A_\varepsilon)$. Moreover, we have that each triangle $T \in \mathcal{T}$ is $(1+\varepsilon)$-bi-Lipschitz equivalent to the triangle $\bar{T}_0$, which implies that

$$(1+\varepsilon)^{-2}|A| \leq |A_\varepsilon| \leq (1+\varepsilon)^2|A|.$$

Letting $\varepsilon \to 0$, we have that $\text{Area}(A_\varepsilon)$ converges to $\text{Area}(A)$; see Theorem VIII.2 in [2]. From this we conclude that $|A| = \text{Area}(A)$, which completes the proof.

7. Approximation by smooth Riemannian surfaces

In this section, we prove that every $\text{CAT}(\kappa)$ surface, $\kappa \in \mathbb{R}$, is the uniform limit of smooth Riemannian surfaces with Gaussian curvature bounded above by κ and having locally uniformly bounded area. In Section 5.3, we showed that such a surface is the uniform limit of $\text{CAT}(\kappa)$ model polyhedral surfaces. What remains is to show that one can smoothen out the vertices while retaining the $\text{CAT}(\kappa)$ condition.

Let X be a polyhedron with faces of constant curvature κ. The metric in a neighborhood of a vertex, in polar coordinates, is defined by the conformal length element $ds^2 = \lambda(r)(dr^2 + r^2 d\theta^2)$, where

$$\lambda(r) = \frac{4\alpha^2 r^{2(\alpha-1)}}{(1+\kappa \cdot r^{2\alpha})^2}. \tag{7.1}$$

and $\alpha \geqslant 1$ is the total angle at the vertex divided by 2π. If $\alpha = 1$, then we just have the usual constant-curvature metric. Observe that $\lim_{r\to 0} \lambda(r) = 0$ if $\alpha > 1$. The Gaussian curvature of this metric outside of the vertex is given by the formula

$$K(r) = \frac{-1}{2\lambda(r)}\left(\frac{\partial^2}{\partial r^2}\log(\lambda(r)) + \frac{1}{r}\frac{\partial}{\partial r}\log(\lambda(r))\right).$$

We have the following main lemma.

Lemma 7.1. *Consider the Euclidean ball $B_{\text{Euc}}(\mathbf{0}, R) \subset \mathbb{R}^2$ for some $R > 0$, equipped with the metric $ds^2 = \lambda(r)(dr^2 + r^2 d\theta^2)$, with λ given by (7.1), with $\alpha > 1$ and $\kappa \geq 0$. For all $\delta \in (0, R)$ sufficiently small, there is a smooth Riemannian metric $ds_\delta^2 = \lambda_\delta(r)(dr^2 + r^2 d\theta^2)$ that agrees with ds^2 on $B_{\text{Euc}}(\mathbf{0}, R)$ outside the Euclidean ball $B_{\text{Euc}}(\mathbf{0}, \delta)$ and has Gaussian curvature at most κ. Moreover, λ_δ is an increasing function on $B_{\text{Euc}}(\mathbf{0}, \delta)$.*

Proof. We give two approaches to proving this lemma: the first based on interpolating with a flat surface and the second based on interpolating with a hyperbolic surface. The first approach is simpler but only covers the case where $\kappa \geq 0$.

1. Interpolating with a flat metric. We assume here that $\kappa \geq 0$. We define the smoothened metric λ_δ as follows. First, we define for each $0 < a < b$ the smooth cutoff function $\varphi_{a,b} : [0, \infty) \to [0, 1]$ by

$$\varphi_{a,b}(r) = \begin{cases} 0 & \text{if } r \leq a \\ \frac{e^{-1/(r-a)}}{e^{-1/(b-r)} + e^{-1/(r-a)}} & \text{if } a \leq r \leq b \\ 1 & \text{if } r \geq b \end{cases}$$

Write $\varphi_b = \varphi_{b/2,b}$. Next, let

$$g_\delta(r) = \left(\frac{\partial}{\partial r}\log(\lambda(r))\right)\varphi_\delta(r)$$

and

$$\lambda_\delta(r) = \lambda(\delta)\exp\left(\int_\delta^r g_\delta(t)\,dt\right).$$

We claim that this choice of λ_δ gives the required properties. It is easy to see that $\lambda_\delta(r) = \lambda(r)$ whenever $r \geq \delta$. Moreover, $g_\delta(r) = 0$ if $r \leq \delta/2$, from which it follows that λ_δ is constant on the interval $[0, \delta]$. From this, we see that ds_δ is locally isometric to the plane on $B(\mathbf{0}, \delta)$. We conclude that ds_δ is a smooth Riemannian metric. Observe now that

$$\frac{\partial}{\partial r}\log(\lambda(r)) = \frac{-2 + 2\alpha - 2\kappa \cdot r^{2\alpha}(1+\alpha)}{r(1 + \kappa \cdot r^{2\alpha})}, \tag{7.2}$$

which is positive provided that $\alpha > 0$ and r is sufficiently small depending on α. Take $\delta > 0$ small enough so that the quantity in (7.2) is positive for all $r \in (0, \delta)$. It follows that $g_\delta(r)$ is non-negative for all $r \in (0, \delta)$ and thus that $\lambda_\delta(r)$ is increasing on this interval. We also choose δ small enough so that $\lambda(\delta) \leq 1$.

The curvature of the metric ds_δ^2 is

$$\begin{aligned} K_\delta(r) &= \frac{-1}{2\lambda_\delta(r)}\left(g_\delta'(r) + \frac{g_\delta(r)}{r}\right) \\ &= \frac{-1}{2\lambda_\delta(r)}\left(\left(\frac{\partial}{\partial r}\log(\lambda(r))\right)\varphi_\delta'(r) + \left(\frac{\partial^2}{\partial^2 r}\log(\lambda(r))\right)\varphi_\delta(r) + \frac{\left(\frac{\partial}{\partial r}\log(\lambda(r))\right)\varphi_\delta(r)}{r}\right). \end{aligned}$$

To complete the proof, we must show that $K_\delta(r) \leq \kappa$. This is equivalent to the inequality

$$-g_\delta'(r) - \frac{g_\delta(r)}{r} \leq 2\kappa \cdot \lambda_\delta(r), \tag{7.3}$$

which we can write as

$$-\left(\frac{\partial}{\partial r}\log(\lambda(r))\right)\varphi_\delta'(r) - \left(\frac{\partial^2}{\partial^2 r}\log(\lambda(r))\right)\varphi_\delta(r) - \frac{\left(\frac{\partial}{\partial r}\log(\lambda(r))\right)\varphi_\delta(r)}{r} \leq 2\kappa\cdot\lambda(\delta)\exp\left(\int_\delta^r g_\delta(t)\,dt\right).$$

To handle the first term on the left-hand side of this equation, observe that $\left(\frac{\partial}{\partial r}\log(\lambda(r))\right)\varphi_\delta'(r) \geq 0$, again using the property that $\frac{\partial}{\partial r}\log(\lambda(r)) > 0$ for all $r \in (0, \delta)$. The other two terms can be written as

$$\left(-\frac{\partial^2}{\partial^2 r}\log(\lambda(r)) - \frac{\left(\frac{\partial}{\partial r}\log(\lambda(r))\right)}{r}\right)\varphi_\delta(r) = 2\kappa \cdot \lambda(r)\varphi_\delta(r).$$

Thus, to verify (7.3), it is enough to show that $2\kappa \cdot \lambda(r)\varphi_\delta(r) \leq 2\kappa \cdot \lambda_\delta(r)$ for all $r \in (\delta/2, \delta)$. This is trivially satisfied if $\kappa = 0$. If $\kappa > 0$, this can be written as

$$\lambda(r)\varphi_\delta(r) \leq \lambda(\delta)\exp\left(\int_\delta^r g_\delta(t)\,dt\right). \tag{7.4}$$

First, since $\lambda(r) \leq \lambda(\delta) \leq 1$ and $\varphi_\delta(r) \leq 1$ for all $r \in (0, \delta)$, we have

$$\lambda(r)\varphi_\delta(r) \leq \lambda(r) \leq \lambda(r)^{\varphi_\delta(r)} = \exp\left((\log\lambda(r))\varphi_\delta(r)\right).$$

Then, using the properties that $r < \delta$ and that φ_δ is increasing and bounded by 1, we obtain

$$\begin{aligned}\exp\left((\log \lambda(r))\varphi_\delta(r)\right) &= \exp\left(\left(\log \lambda(\delta) + \int_\delta^r \frac{\partial}{\partial t} \log \lambda(t)\, dt\right) \varphi_\delta(r)\right)\\ &\leq \exp\left(\left(\log \lambda(\delta) + \int_\delta^r \frac{\partial}{\partial t} \log \lambda(t) \varphi_\delta(t)\, dt\right)\right)\\ &= \lambda(\delta) \exp\left(\int_\delta^r g_\delta(t)\, dt\right).\end{aligned}$$

This concludes our verification that $K_\delta(r) \leq \kappa$.

2. Interpolating with the hyperbolic metric. The following proof handles the general case. It is similar to the previous one, so we focus on the differences. By rescaling, we may assume that $\kappa \geq -1$. Let $\bar{\lambda}(r) = 4/(1-r^2)^2$ be the conformal weight for the hyperbolic metric, which is defined for all $r \in (0,1)$. We make two immediate observations. First, since $\alpha > 1$ and $\kappa \geq -1$, we have $\bar{\lambda}(r) \geq \lambda(r)$ for all $r \in (0,1)$. Second, the inequality

$$\frac{\partial}{\partial r} \log(\lambda(r)) \geq \frac{\partial}{\partial r} \log\left(\bar{\lambda}(r)\right) \tag{7.5}$$

holds for all $r > 0$ sufficiently small. Assume then that $\delta > 0$ is small enough that (7.5) holds for all $r \leq \delta$. We also necessarily assume that $\delta < 1$.

Our main modification is to redefine g_δ above using the formula

$$g_\delta(r) = \left(\frac{\partial}{\partial r} \log(\lambda(r))\right) \varphi_\delta(r) + \left(\frac{\partial}{\partial r} \log\left(\bar{\lambda}(r)\right)\right)(1 - \varphi_\delta(r)).$$

Keep the same definition for λ_δ, except that we use the new formula for g_δ. The conformal weight λ_δ on $B_{\mathrm{Euc}}(\mathbf{0}, \delta/2)$ is a rescaling of the hyperbolic metric by the factor $\lambda(\delta) \exp\left(\int_\delta^{\delta/2} g_\delta(t)\, dt\right) \leq \lambda(\delta/2)$, which is less than 1 since $\delta < 1$. It follows that $K_\delta(r) \leq \kappa$ whenever $r \leq \delta/2$.

Next, to verify (7.3), there are two details to check. First, we observe from (7.5) that

$$\left(\frac{\partial}{\partial r} \log(\lambda(r)) - \frac{\partial}{\partial r} \log\left(\bar{\lambda}(r)\right)\right) \varphi_\delta'(r) \geq 0$$

for all $r \in (0, \delta)$. Then, instead of (7.4) we must check that

$$\kappa \cdot \lambda(r)\varphi_\delta(r) - \bar{\lambda}(r)(1 - \varphi_\delta(r)) \leq \kappa \cdot \lambda_\delta(r)$$

for all $r \in (0, \delta)$. This is trivial if $\kappa = 0$. If $\kappa > 0$, it suffices to show that

$$\cdot \lambda(r)\varphi_\delta(r) \leq \cdot \lambda_\delta(r),$$

which can be done similarly to the first part of the proof. In the case where $\kappa < 0$, it suffices to show that

$$\lambda(r)\varphi_\delta(r) + \bar{\lambda}(r)(1 - \varphi_\delta(r)) \geq \lambda(\delta) \exp\left(\int_\delta^r g_\delta(t)\right)$$

Note that, since $\kappa < 0$, this inequality is reversed relative to (7.4). We first use Young's inequality to obtain

$$\lambda(r)\varphi_\delta(r) + \bar{\lambda}(r)(1 - \varphi_\delta(r)) \geq \exp\left((\log \lambda(r))\varphi_\delta(r) + (\log \bar{\lambda}(r))(1 - \varphi_\delta(r))\right).$$

The right-hand side can be written as

$$\exp\left(\left(\log \lambda(\delta) + \int_\delta^r \frac{\partial}{\partial t} \log(\lambda(t))\, dt\right) \varphi_\delta(r) + \left(\log \bar{\lambda}(\delta) + \int_\delta^r \frac{\partial}{\partial t} \log\left(\bar{\lambda}(t)\right) dt\right)(1 - \varphi_\delta(r))\right).$$

Since $\bar{\lambda}(r) \geq \lambda(r)$ and φ_δ is increasing, this quantity is greater than

$$\exp\left(\log\lambda(\delta) + \left(\int_\delta^r \frac{\partial}{\partial t}\log(\lambda(t))\varphi_\delta(t)\,dt\right) + \left(\int_\delta^r \frac{\partial}{\partial t}\log\big(\bar{\lambda}(t)\big)(1-\varphi_\delta(t))\,dt\right)\right)$$
$$=\lambda(\delta)\exp\left(\int_\delta^r g_\delta(t)\,dt\right).$$

This concludes the proof. □

We now proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4. Let X be a CAT(κ) surface, where $\kappa \geq 0$. Let (ε_n) be a sequence of positive reals limiting to 0, and X_n the approximating model polyhedral surface given in Section 5.3 for the value ε_n. In particular, there is a homeomorphic ε_n-isometry $\psi_n\colon X_n \to X$ with the property that $|\psi_n^{-1}(A)|$ converges to $|A|$ for all sets $A \subset X$. For each vertex v_i of X_n, $i \in \mathbb{N}$, let $r_i > 0$ be such that $B(v_i, 2r_i)$ does not contain any other vertex of X_n. Note that this implies that that the balls $B(v_i, r_i)$ are disjoint. We also require that $\sum_{i=1}^\infty r_i < \varepsilon_n$. The ball $B(v_i, r_i)$ is locally isometric to $B_{\mathrm{Euc}}(\mathbf{0}, R_i)$ for some $R_i > 0$, equipped with the metric given in Lemma 7.1. Choose $\delta_i > 0$ sufficiently small so that $B_{\mathrm{Euc}}(\mathbf{0}, \delta_i)$ has diameter and area at most $\varepsilon_n 2^{-i}$, which can be done since λ_δ is increasing for $r \in (0, \delta)$ and $\lambda(\delta) \to 0$ as $\delta \to 0$. Let Y_n be the smooth Riemannian surface obtained by applying Lemma 7.1 to each neighborhood $B_{\mathrm{Euc}}(\mathbf{0}, \delta_i)$. Let V_i denote the neighborhood in X_n on which the metric is altered.

Define a map φ_n from X_n to Y_n by mapping V_i to the corresponding neighborhood of Y_n in a continuous way fixing ∂V_i and as the identity map otherwise. By construction, φ_n is a $(2\varepsilon_n)$-isometry. Moreover, for any set $A \subset X_n$, we have $|\varphi_n(A)| \leq |A| + \varepsilon_n$. Then $\psi_n \circ \varphi_n^{-1}$ is a (3ε)-isometry from Y_n to X with the property that $|\varphi_n \circ \psi_n^{-1}(A)|$ converges to A for all sets $A \subset X$. □

Remark 7.2. The ideas of Lemma 7.1 also apply to smoothing vertices of model polyhedral surfaces with curvature bounded below. Assumes that X is a model polyhedral surface with curvature bounded below for some $\kappa \leq 0$. Then the metric is given in a neighborhood of a vertex by (7.1), where $\alpha < 1$. Thus, for sufficiently small r, one has $\frac{\partial}{\partial r}\log(\lambda(r)) > 0$ by (7.2). Since $\kappa \leq 0$, the chain of inequalities at the end of the first part of the proof of Lemma 7.1 establishes (7.3) with "$\leq$" replaced by "$\geq$". We conclude that the Riemannian approximation in Lemma 7.1 has curvature bounded below by κ.

The general case can be handled by interpolating with the spherical metric, by modifying the above argument for the hyperbolic metric. We leave the details to the interested reader.

8. Excess-curvature inequality

In this section, we give a proof of Theorem 1.2 relating the excess and total curvature of an arbitrary CAT(κ) triangle. Let T be a triangle with angles α, β, γ. By subdividing if necessary, we may assume by Lemma 3.5 that T is convex. In particular, we can now consider T as its own metric space. Let T_n be a sequence of model polyhedral approximations converging to T as constructed in the third proof of Theorem 1.1; such a sequence can also be found by invoking the general theory of surfaces of bounded curvature. Note that T_n is also a triangle, since the edges of T are still geodesics in the polyhedral approximation. Let $\alpha_n, \beta_n, \gamma_n$ denote the angles in T_n corresponding to the same edges in T as α, β, γ, respectively. The approximations T_n come with a Lipschitz map $\varphi_n\colon T_n \to T$. This implies that $\alpha \leq \alpha_n$, $\beta \leq \beta_n$, $\gamma \leq \gamma_n$, and thus that $\delta(T) \leq \delta(T_n)$.

The rotation of the boundary ∂T_n is

$$\tau(T_n) = (\pi - \alpha_n) + (\pi - \beta_n) + (\pi - \gamma_n) + \tau_0(T_n) = 2\pi - \delta(T_n) + \tau_0(T_n),$$

where $\tau_0(T_n)$ is the contribution to the rotation of ∂T_n from the interior points of the edges. The fact that each edge is a geodesic implies that $\tau_0(T_n) \leq 0$. In particular, $\delta(T_n) \leq 2\pi - \tau(T_n)$. The

Gauss–Bonnet theorem for surfaces of bounded curvature states that $\tau(T_n) + \omega(T_n) = 2\pi$, where $\omega(T_n)$ is the interior curvature of T_n. This relation gives

$$\delta(T_n) \leq 2\pi - \tau(T_n) = \omega(T_n).$$

Now, the positive part $\omega_+(T_n)$ satisfies $\omega_+(T_n) = \kappa|T_n|$. Since $\omega(T_n) \leq \omega_+(T_n)$, we conclude that

$$\delta(T) \leq \delta(T_n) \leq \kappa|T_n|.$$

Finally, the sequence $|T_n|$ converges to $|T|$ as $n \to \infty$. This completes the proof.

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