A Description of Free Homotopy Classes of Paths on Surfaces with a Negative Euler Characteristic Using Pant Decompositions

Sacha Caron

August 30, 2016

Abstract

With the exception of the sphere, disk, annulus and torus, all surfaces yield a pant decomposition that, with additional structure, describes free homotopy classes of closed and boundary fixed paths on that surface. This allows algebraic manipulation of a pant decomposition and its free homotopy classes.

1 Cell Complex Descriptions of Paths

Given any compact orientable surface S there's a homeomorphism $H : S \longrightarrow C_S$, where C_S is a cell complex, that partitions S into simply connected subspaces.

Let $\phi : \Delta \to C_S$, where $\Delta = \{a_1, a_1^{-1}, a_2, a_2^{-1}, ..., a_k, a_k^{-1}\}$ is an alphabet with k the number of 1-cells in C_S, mark each 1-cell ϵ_i of C_S with their respective pair of letters (a_i, a_i^{-1}). Each letter represents a unique way to transversely cross that 1-cell.

A path $\varphi : [0,1] \longrightarrow C_S$ that doesn't intersect 0-cells and intersects 1-cells transversely can be described by a word ω. Starting at $\varphi(0)$ and ending at $\varphi(1)$, a word ω is generated by its oriented intersections with the 1-cells.

Minimal A path φ is minimal if $|\omega| \leq |\omega'|$ where $|\omega|$ is the number of letters in ω and ω' is all paths φ' homotopic to φ.

In this paper we'll restrict our attention to paths φ that are minimal, with boundaries mapped to 1-cells, that don't intersect 0-cells and that intersect 1-cells transversely. Also, we'll restrict our attention to paths with $|\omega| > 1$.

It's important to note that paths with different orientations aren't homotopic.

Proposition 1.1: If $\omega_1 = \omega_2$ then φ_1 and φ_2 are free homotopic.

Proof: If $\omega_1 = \omega_2$ then φ_1 and φ_2 intersect the same 1-cells in the same order. Let $h_{1/2}$ be a homotopy that maps $\varphi_2 \cap \epsilon_i$ to $\varphi_1 \cap \epsilon_i$ at each ϵ_i. Since $C_S \setminus C_{S,1}$ is a set of simply connected subspaces, then there exists a homotopy $h_{2/2}$ from each subpath $\varphi_{2,i}$ to each subpath $\varphi_{1,i}$. That said, $h_{2/2}(h_{1/2})$ is a homotopy from φ_2 to φ_1 and φ_1 and φ_2 are homotopic. Ξ

Now we're able to describe paths on compact orientable surfaces and we know that if two paths are described by the same word then they're homotopic. However, we haven't proven that the converse is true.

2 Pant Compositions, Pants with 3 Seams, and Boundary Fixed Paths

With the exception of the sphere, disk, annulus and torus, it can be said that a compact orientable surface S yields a pant decomposition Π. Π made of pants π with 2 seams can describe free homotopy classes but π with 3 seams deserves attention for two reasons. First, the 3 seam structure is the only other seam structure on π that describes the classes where there seams aren't redundant. Second, the 3 seam structure has greater symmetry which simplifies cutting, gluing, and transformations under A-moves and S-moves.

Let π be the diagram in figure 1 marked by $\Delta = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \gamma^{-1}$ $, a, a^{-1}, b, b^{-1}, c, c^{-1}, A, A^{-1}, B, B^{-1}, C, C^{-1}\}$.

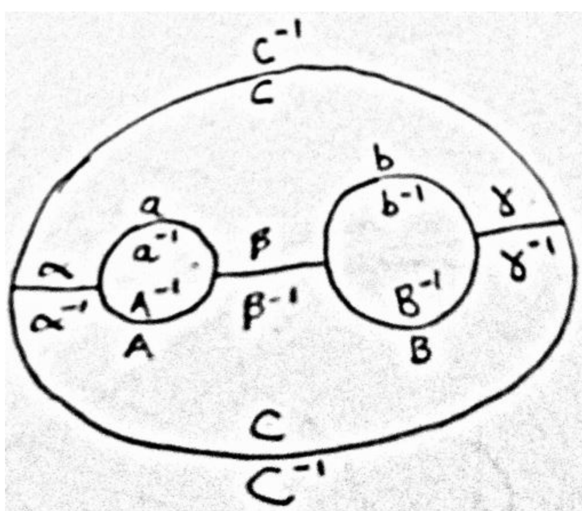

Figure 1: Labeled Pair of Pants w/ 3 Seams

Decomposing a surface into pairs of pants decomposes the paths into closed paths on a pair of pants and boundary fixed paths on pairs of pants.

Boundary Fixed Path A boundary fixed path φ is a path on a surface S such that $\varphi(0)$ and $\varphi(1)$ are boundary points of S.

Also, we get two homotopy equivalence relations at the gluing boundaries.

Boundary Slide Let gG and $g'G'$ be two different boundaries of either the same or different pairs of pants. Also, let (α, α^{-1}) label the seam intersecting gG and (α', α'^{-1}) label the seam intersecting $g'G'$ such that the two seams share a 0-cell. A boundary slide between minimal paths φ_1 and φ_2 with descriptions ω_1 and ω_2 is a homotopy described as $\alpha G \longrightarrow g\alpha'$.

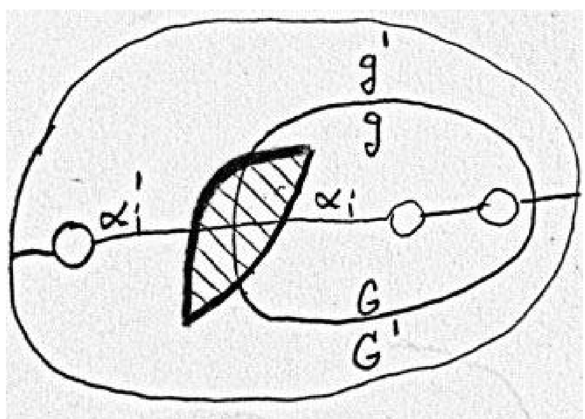

Figure 2: Boundary Slide

Seam Slide A seam slide is a homotopy between φ_1 and φ_2 where φ_1 and φ_2 are minimal paths on a surface S that are homotopic to the same seam. This homotopy is described by $AgA' \longrightarrow aGa'$ and $aa \longrightarrow AA$.

We'll use boundary and seam slide equivalence relations to simplify our descriptions. That said, if ω_1 and ω_2 are equivalent after a finite series of boundary slides and seam slides then $\omega_1 = \omega_2$.

Proposition 2.1 Let φ and φ' be minimal closed or boundary fixed paths on π_i. $\varphi \sim \varphi'$ iff $\omega = \omega'$.

Proof: If $\varphi \sim \varphi'$ it's implied by definition that there exists a homotopy $h(x,t)$ between φ and φ'. Suppose $\omega \neq \omega'$ then at some t φ intersecst 1-cell ϵ and φ' intersects 1-cell ϵ' where $\epsilon \neq \epsilon'$. At t, either φ and φ' intersect before and after ϵ and ϵ' or before or after and at ϵ and ϵ'. In the first case ϵ and ϵ' are seams and

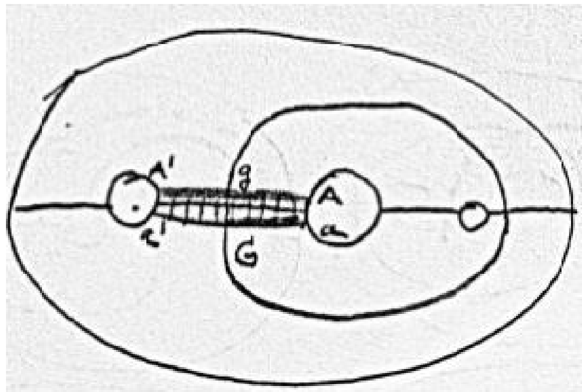

Figure 3: Seam Slide

the region in between φ and φ' isn't simply connected which implies h doesn't exist which is a contradiction. In the second case, the boundary letters at ϵ and ϵ' must be the same letter but in uppercase and lowercase. If the letters before or after the boundary letters are seam letters then either one path isn't minimal or they're not homotopic. If they're boundary letters then there's a seam slide equivalence. That said $\varphi \sim \varphi'$ implies $\omega = \omega'$.

If $\omega = \omega'$ then by proposition 1.1 it's implied that $\varphi \sim \varphi'$. Ξ

If φ is a closed path on the pair of pants then it's unchanged when we glue boundaries together. So, from this point forward we'll focus our attention on boundary fixed paths. When gluing boundaries together we'll say that if two paths fixed at different boundaries end at the same 1-cell then they can be connected.

Proposition 2.2 Let φ_1 and φ_2 be minimal boundary fixed paths on Π. If Π glues the end points of φ_1 and φ_2 together, then the new path φ_3 on Π is uniquely minimal or minimal under boundary slide and therefore $\varphi_3 \sim \varphi_4$ iff $\omega_3 = \omega_4$.

Proof: Since φ_1 and φ_2 are minimal then the letters preceding the glued boundary letter are either boundary letters of the other two boundaries or seam letters of the seam connecting the other two boundaries. If they're both seam letters then the path is uniquely minimal. If one is a seam letter and the other is a boundary the same can be said. If they're both boundary letters of boundaries which aren't connected by a combined seam then it's uniquely minimal. If they're both boundary letters of boundaries connected by a combined seam then there's a boundary slide equivalence. That said, if $\varphi_3 \sim \varphi_4$ then $\omega_3 = \omega_4$.

If $\omega_3 = \omega_4$ then by proposition 1.1 it's implied that $\varphi_3 \sim \varphi_4$. Ξ

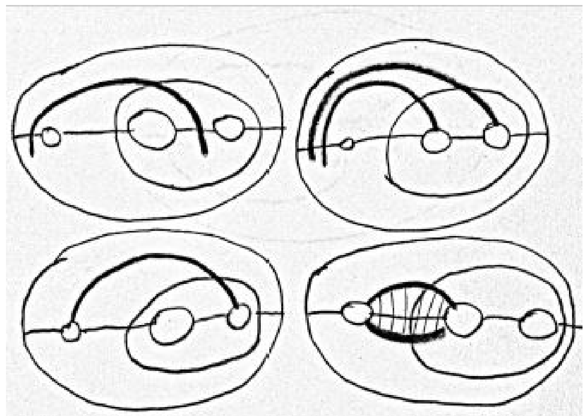

Figure 4: Five Ways of Connecting Minimal Boundary Fixed Paths

In order to be able to describe all free homotopy classes on a pant composition we need to introduce a couple more definitions.

Winding Minimal Given a boundary fixed path φ, a wind $\frac{n}{2}$ where $n \in Z$, on φ is a homotopy to a path φ' that is a map which moves the boundary fixed point of φ around the boundary by πn. It's described by a n letter reduced subword generated from the subalphabet of seam letters from seams that touch the gluing boundary. A path is winding minimal if it's minimal with the exception of a wind.

Unwinding An unwinding is the minimalization of a path φ_1, which is the result of a pant gluing, by using boundary slides to create reducable subwords in ω_1 therefore shortening it.

Proposition 2.3 Let φ_1 be a boundary fixed path on π_1 and φ_2 be a boundary fixed path on π_2 where at least one is winding minimal and the other minimal or winding minimal. If Π glues the end points of φ_1 and φ_2 together into φ_3 then $\varphi_3 \sim \varphi_4$ iff $\omega_3 = \omega_4$.

Proof: If φ_1 and φ_2 are glued together and both of them are winding minimal, we can use boundary slides and winding cancellations to make φ_1 winding minimal and φ_2 minimal. If φ_2 is coming from a seam then the only minimal paths φ_4 homotopic to φ_3 are those obtained through boundary slides. If φ_2 is coming from a boundary and φ_1 is coming from a seam which doesn't connect to that boundary then φ_4 is equivalence under boundary slides. If φ_1 is coming from a seam which does connect to that boundary then φ_3 isn't minimal and the boundary letter is switched to its adjacent one, the gluing boundary is switched to the adjacent one, and the seam letter next to it disappears. After that φ_1 is either coming from the middles seam, a boundary, or more wind. If it's from the middle seam then clearly φ_3 is minimal. If it's from a boundary connected to it by seam then the equivalences are seam slides. If it's from the other boundary it's minimal. If it's from a wind then it's minimal under boundary slides. Therefore, if $\varphi_3 \sim \varphi_4$ then $\omega_3 = \omega_4$.

If $\omega_3 = \omega 4$ then by Proposition 1.1 $\varphi_3 \sim \varphi_4$. Ξ.

We now have the converse of proposition 1.1 proved on any pant composition with the 3 seam structure if the paths φ are compositions of minimal closed, minimal boundary fixed, and/or winding minimal boundary fixed paths. The last step is to prove that if φ is a minimal path on a pant composition and it's a composition of those three types of subpaths.

Proposition 2.4 If φ is a minimal closed or boundary fixed path on Π then it can be described as a composition of minimal or winding minimal closed or boundary fixed paths on a collection of pants of Π.

Proof: Since φ is a minimal closed or boundary fixed path on Π then once Π is decomposed it's either a closed minimal path on a pant π, a minimal boundary fixed path on a pant π, or a composition of boundary fixed paths on subpant composition Π_1. If it's a closed minimal path or a minimal boundary fixed path on a pant π then clearly it can be described as a composition of minimal closed or boundary fixed paths. Suppose that in the third case the boundary fixed paths are neither minimal nor winding minimal, then at least one subpath φ_i isn't minimal or winding minimal. φ_i will be described by ω_i which starts and ends with boundary letters. On either side, since it can't be minimal, the next letters in are either winding letters or the opposing seam. In between those is a subword of seam letters without inverse pairs. That said, it must be either winding minimal or minimal which is a contradiction and therefor φ can be described as a composition of minimal or winding minimal closed or boundary fixed paths. Ξ

It's now been shown that a 3 seam pant decomposition of a compact orientable surface along with the boundary slide and seam slide equivalence relations makes it so that $\varphi \sim \varphi'$ iff $\omega = \omega'$.