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ORIGINAL PAPER

Minimal intersection of curves on surfaces

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Abstract This paper is a consequence of the close connection between combinatorial group theory and the topology of surfaces. In the eighties Goldman discovered a Lie algebra structure on the vector space generated by the free homotopy classes of oriented curves on an oriented surface. The Lie bracket [a, b] is defined as the signed sum over the intersection points of a and b of their loop product at the intersection points. If one of the classes has a simple representative we give a combinatorial group theory description of the terms of the Lie bracket and prove that this bracket has as many terms, counted with multiplicity, as the minimal number of intersection points of a and b. In other words the bracket with a simple element has no cancellation and determines minimal intersection numbers. We show that analogous results hold for the Lie bracket (also discovered by Goldman) of unoriented curves. We give three applications: a factorization of Thurston's map defining the boundary of Teichmüller space, various decompositions of the underlying vector space of conjugacy classes into ad invariant subspaces and a connection between bijections of the set of conjugacy classes of curves on a surface preserving the Goldman bracket and the mapping class group.

Keywords Surfaces · Simple closed curve · Minimal intersection number · Fundamental group · Conjugacy classes · Amalgamated free products · HNN extensions

Mathematics Subject Classification (2000) 57M99 · 20E06

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1 Introduction

Let a and b denote isotopy classes of embedded closed curves on a surface Σ. Denote by $i(a, b)$ the minimal possible number of intersection points of curves representing a and b, where the intersections are counted with multiplicity. The function $i(a, b)$ plays a central role in Thurston's work on low dimensional topology (see, for instance, [12,14,29]). Let $[a, b]$ denote the Lie bracket on the vector space of the free homotopy classes of all essential directed closed curves on Σ. This Lie bracket originated from Wolpert's cosine formula, Thurston's earthquakes in Teichmüller space and Goldman's study of Poisson brackets (see [13]). The Lie algebra of oriented curves has been generalized using the loop product to a string bracket on the reduced $\mathbb{S}^1$-equivariant homology of the free loop space of any oriented manifold. (see [6,7].)

The main purpose of this paper is to prove that both, the Goldman Lie bracket on oriented curves and the Goldman Lie bracket on unoriented curves "count" the minimal number of intersection points of two simple curves.

In order to give a more precise statement of our results, let us review the definition of the Lie algebra for oriented curves discovered by Goldman: given two such free homotopy classes of directed curves on an orientable surface, take two representatives intersecting each other only in transverse double points. Each one of the intersection points will contribute a term to the geometric formula for the bracket. Each of these terms is defined as follows: take the conjugacy class of the curve obtained by multiplying the two curves at the intersection point and adjoin a negative sign if the orientation given by the ordered tangents at that point is different from the orientation of the surface. The bracket on W, the vector space of free homotopy classes is the bilinear extension of this construction. It is remarkable that this construction is well defined and satisfies skew-symmetry and Jacobi.

The bracket of unoriented curves can be defined on the subspace V of W fixed by the operation of reversing direction, as the restriction of the bracket. The subspace V is generated by elements of W of the form $a + \overline{a}$ where a is a basis element and where $\overline{a}$ denotes a with opposite direction. Let us identify unoriented curves up to homotopy with these expressions $a+\overline{a}$. The Lie bracket of two unoriented curves, $a+\overline{a}$ and $b+\overline{b}$, is then defined geometrically by $([a, b] + \overline{[a, b]}) + ([a, \overline{b}] + \overline{[a, \overline{b}]})$, which equals $[a + \overline{a}, b + \overline{b}]$ using $\overline{[a, b]} = [\overline{a}, \overline{b}]$. An *unoriented term* is a term of the form $c(z + \overline{z})$, where c is an integer and z is a conjugacy class, that is, an element of the basis of V multiplied by an integer coefficient.

Since this Lie bracket uses the intersection points of curves, a natural problem was to study how well it reflects the intersection structure. In this regard, Goldman [13] proved the following result.

Goldman's Theorem *If the bracket of two free homotopy classes of curves (oriented or unoriented) is zero, and one of them has a simple representative, then the two classes have disjoint representatives.*

Goldman's proof uses the Kerckhoff earthquake convexity property of Teichmüller space [19] and in [13] he wondered whether this topological result had a topological proof. In [3] we gave such a proof when the free homotopy class contains a non-separating simple closed curve and the surface has non-empty boundary. Here we will give combinatorial proof of our Main Theorem (see below), which a generalization of Goldman's result.

Now suppose that a can be represented by a simple closed curve α. Then for any free homotopy class b there exists a representative that can be written as a certain product which involves a sequence of elements in the fundamental group or groups of the connected components of $\Sigma \setminus \alpha$ (see Sects. 2, 4 for precise definitions.) The number of terms of the sequence for b with respect to a is denoted by $t(a, b)$.

Main Theorem *Let a and b be two free homotopy classes of directed curves on an orientable surface. If a contains a simple closed curve then the following nonnegative integers are equal:*

- (i) *The number of terms, counted with multiplicity, of the Goldman Lie bracket for oriented curves* $[a, b]$.
- (ii) *The number divided by two of unoriented terms (of the form* $x + \overline{x}$*), counted with multiplicity, of the Goldman Lie bracket for unoriented curves* $[a + \overline{a}, b + \overline{b}]$.
- (iii) *The minimal number of intersection points of a and b,* $i(a, b)$.
- (iv) *The number of terms of the sequence for b with respect to a,* $t(a, b)$.

We obtain the following global characterization of conjugacy classes containing simple representatives.

Corollary of the Main Theorem *Let a denote a free homotopy class of curves on a surface. Then the following statements are equivalent.*

1. *The class a has a representative that is a power of a simple curve.*
2. *For every free homotopy class b the (oriented) bracket* $[a, b]$ *has many as oriented terms counted with multiplicity as the minimal intersection number of a and b.*
3. *For every free homotopy class b the (unoriented) bracket* $[a, b]$ *has many as unoriented terms counted with multiplicity as twice the minimal intersection number of a and b.*

In [5] a local characterization of simple closed curves in terms of the Lie bracket will be given. Actually, the problem of characterizing algebraically embedded conjugacy classes was the original motivation of [3,6,7].

Here is a brief outline of the arguments we follow to prove the main theorem.

1. The key point is that when a curve is simple, we can apply either HNN extensions or amalgamated products to write elements of the fundamental group of the surface as a product that involves certain sequences of elements of subgroups which are the fundamental group of the connected components of the surface minus the simple curve.
2. Using combinatorial group theory tools we show that if certain equations do not hold in an HNN extension or an amalgamated free product then certain products of the sequences in (1) cannot be conjugate, (Sects. 2 and 4.)
3. We show that each of the terms of the bracket can be obtained by inserting the conjugacy class of the simple curve in different places of the sequences in (1) made circular. (Sects. 3 and 5.)
4. We show that the equations of (2) do not hold in the HNN extensions and amalgamated free products determined by a simple closed curve. (Sect. 6)

The Goldman bracket extends to higher dimensional manifolds as one of the String Topology operations. Abbaspour [1] characterizes hyperbolic three manifolds among closed three manifolds using the loop product which is another String Topology operation. Some of his arguments rely on the decomposition of the fundamental group of a manifold into amalgamated products based on torus submanifolds, and the use of this decomposition to give expressions for certain elements of the fundamental group, which in term, gives a way of computing the loop product.

Here is the organization of this work: in Sect. 2 we list the known results about amalgamated products of groups we will use, and we prove that certain equations do not hold in such groups. In Sect. 3, we apply the results of the previous section to find a combinatorial description of the Goldman bracket of two oriented curves, one of them simple and separating (see Fig. 3.) In Sect. 4 we list results concerning HNN extensions and we prove that certain equations do not hold in such groups. In Sect. 5, we apply the results of Sect. 4 to describe combinatorially the bracket of an oriented non-separating simple closed curve with another oriented curve (see Fig. 5.) In Sect. 6 we prove propositions about the fundamental group of the surface which will be used to show that our sequences satisfy the hypothesis of the Main Theorems of Sects. 2 and 4. In Sect. 7 we put together most of the above results to show that there is no cancellation in the Goldman bracket of two directed curves, provided that one of them is simple. In Sect. 8 we define the bracket of unoriented curves and prove that there is no cancellation if one of the curves is simple. In Sect. 9 we exhibit some examples that show that the hypothesis of one of the curves being simple cannot be dropped (see Fig. 9.) In the next three sections, we exhibit some applications of our main results. More precisely, in Sect. 10, we show how our main theorem yields a factorization of Thurston's map on the set of all simple conjugacy classes of curves on a surface, through the power of the vector space of all conjugacy classes to the set of simple conjugacy classes. In Sect. 11 we exhibit several partitions of the vector space generated by all conjugacy classes, which are invariant under certain Lie algebra operations. We conclude by showing in Sect. 12 that if a permutation of the set of conjugacy classes preserves the bracket and simple closed curves, then is determined by an element of the mapping class group of the surface. We conclude by stating some problems and open questions in Sect. 13.

2 Amalgamated products

This section deals exclusively with results concerning Combinatorial Group Theory. We start by stating definitions and known results about amalgamating free products. Using these tools, we prove the main results of this section, namely, Theorems 2.12 and 2.14. These two theorems state that certain pairs of elements of an amalgamating free product are not conjugate. By Theorem 3.4 if b is an arbitrary conjugacy class and a is a conjugacy class containing a separating simple representative, then the Goldman Lie bracket $[a, b]$ can be written as an algebraic sum $t_1 + t_2 + \cdots + t_n$, with the following property: If there exist two terms terms t_i and t_j that cancel, then the conjugacy classes associated with the terms t_i and t_j both satisfy the hypothesis of Theorem 2.12 or both satisfy the hypothesis of Theorem 2.14. This will show that the terms of the Goldman Lie bracket exhibited in Theorem 3.4 are all distinct.

Alternatively, one could make use of the theory of groups acting on graphs (see, for instance, [10]) to prove Theorems 2.12 and 2.14.

Let C, G and H be groups and let $\varphi\colon C \longrightarrow G$ and $\psi\colon C \longrightarrow H$ be monomorphisms. We denote the *free product of* G *and* H *amalgamating the subgroup* C *(and morphisms* φ *and* ψ*)* by $G *_C H$. This group is defined as the quotient of the free product $G * H$ by the normal subgroup generated by $\varphi(c)\psi(c)^{-1}$ for all $c \in C$. (See [16],[25], [26] or [8] for detailed definitions.)

Since there are canonical injective maps from C, G and H to $G *_C H$, in order to make the notation lighter we will work as if C, G and H were included in $G *_C H$.

Definition 2.1 Let n be a non-negative integer. A finite sequence $(w_1, w_2, \ldots, w_n)$ of elements of $G *_C H$ is *reduced* if the following conditions hold,

(1) Each w_i is in one of the factors, G or H.
(2) For each $i \in \{1, 2, \ldots, n-1\}$, w_i and w_{i+1} are not in the same factor.
(3) If $n = 1$, then w_1 is not the identity.

The case $n = 0$ is included as the empty sequence. Also, if n is larger than one then for each $i \in \{1, 2, \ldots, n\}$, $w_i \notin C$, otherwise, (2) is violated.

The proof of the next theorem can be found in [8,25] or [26].

Theorem 2.2 (1) *Every element of $G *_C H$ can be written as a product $w_1 w_2 \cdots w_n$ where $(w_1, w_2, \ldots, w_n)$ is a reduced sequence.* (2) *If n is a positive integer and $(w_1, w_2, \ldots, w_n)$ is a reduced sequence then the product $w_1 w_2 \cdots w_n$ is not the identity.*

We could not find a proof of the next well known result in the literature, so we include it here.

Theorem 2.3 *If $(w_1, w_2, \ldots, w_n)$ and $(h_1, h_2, \ldots, h_n)$ are reduced sequences such the products $w_1 w_2 \cdots w_n$ and $h_1 h_2, \cdots h_n$ are equal then exists a finite sequence of elements of C, namely $(c_1, c_2, \ldots, c_{n-1})$ such that*

(i) $w_1 = h_1 c_1$,
(ii) $w_n = c_{n-1}^{-1} h_n$,
(iii) *For each $i \in \{2, 3, \ldots, n-1\}$, $w_i = c_{i-1}^{-1} h_i c_i$.*

Proof We can assume $n > 1$. Since the products are equal, $h_n^{-1} h_{n-1}^{-1} \cdots h_1^{-1} w_1 w_2 \cdots w_n$ is the identity. By Theorem 2.2(2), the sequence

$$(h_n^{-1}, h_{n-1}^{-1}, \ldots, h_1^{-1}, w_1, w_2, \ldots, w_n)$$

is not reduced. Since the sequences $(h_n^{-1}, h_{n-1}^{-1}, \ldots, h_1^{-1})$ and $(w_1, w_2, \ldots, w_n)$ are reduced, w_1 and h_1^{-1} belong both to $G \setminus C$ or both to $H \setminus C$. Assume the first possibility holds, that is h_1 and w_1 are in $G \setminus C$, (the second possibility is treated analogously.) Set $c_1 = h_1^{-1} w_1$. By our assumption, $c_1 \in G$. The sequence

$$(h_n^{-1}, h_{n-1}^{-1}, \ldots, h_2^{-1}, c_1, w_2, \ldots, w_n)$$

has product equal to the identity. All the elements of this sequence are alternatively in $G \setminus C$ and $H \setminus C$, with the possible exception of c_1. By Theorem 2.2(2), this sequence is not reduced. Then $c_1 \notin G \setminus C$. Since $c_1 \in G$, $c_1 \in C$.

Now consider the sequence

$$(h_n^{-1}, h_{n-1}^{-1}, \ldots, h_3^{-1}, h_2^{-1} c_1 w_2, w_3, \ldots, w_n)$$

By arguments analogous to those we made before, $h_2^{-1} c_1 w_2 \in C$. Denote $c_2 = h_2^{-1} c_1 w_2$. Thus $w_2 = c_1^{-1} h_2 c_2$.

This shows that we can apply induction to find the sequence $(c_1, c_2, \ldots, c_{n-1})$ claimed in the theorem. □

Definition 2.4 A finite sequence of elements $(w_1, w_2, \ldots, w_n)$ of $G *_C H$ is *cyclically reduced* if every cyclic permutation of $(w_1, w_2, \ldots, w_n)$ is reduced.

Notation 2.5 When dealing with free products with amalgamation, every time we consider a sequence of the form $(w_1, w_2, \ldots, w_n)$ we take subindexes mod n in the following way: For each $j \in \mathbb{Z}$, by w_j we will denote w_i where i is the only integer in $\{1, 2, \ldots, n\}$ such that n divides $i - j$.

Remark 2.6 If $(w_1, w_2, \ldots, w_n)$ is a cyclically reduced sequence and $n \neq 1$ then n is even. Also, for every pair of integers i and j, w_i and w_j are both in G or both in H if and only if i and j have the same parity.

The following result is a direct consequence of Theorem 2.2(1) and [26, Theorem 4.6].

Theorem 2.7 *Let s be a conjugacy class of $G *_C H$. Then there exists a cyclically reduced sequence such that the product is a representative of s. Moreover every cyclically reduced sequence with product in s has the same number of terms.*

The following well-known result gives necessary conditions for two cyclically reduced sequences to be conjugate.

Theorem 2.8 *Let $n \geq 2$ and let $(w_1, w_2, \ldots, w_n)$ and $(v_1, v_2, \ldots, v_n)$ be cyclically reduced sequences such that the products $w_1 w_2, \cdots w_n$ and $v_1 v_2 \cdots v_n$ are conjugate. Then there exists an integer $k \in \{0, 1, \ldots, n-1\}$ and a sequence of elements of C, $c_1, c_2, \ldots, c_n$ such that for each $i \in \{1, 2, \ldots, n\}$, $w_i = c_{i+k-1}^{-1} v_{i+k} c_{i+k}$. In particular, for each $i \in \{1, 2, \ldots, n\}$, w_i and v_{i+k} are both in G or both in H.*

Proof By [25, Theorem 2.8], there exist $k \in \{0, 1, 2, \ldots, n-1\}$ and an element c in the amalgamating group C such that

$$w_1 w_2 \cdots w_n = c^{-1} v_{k+1} v_{k+2} \cdots v_{k+n-1} v_{k+n} c.$$

The sequences $(w_1, w_2, \ldots, w_n)$ and $(c^{-1} v_{k+1}, v_{k+2}, \ldots, v_{k+n-1}, v_{k+n} c)$ are reduced and have the same product. By Theorem 2.3 there exists a sequence of elements of C, $(c_1, c_2, \ldots, c_{n-1})$ such that

(i) $w_1 = c^{-1} v_{k+1} c_1$,
(ii) $w_n = c_{n-1}^{-1} v_{k+n} c$,
(iii) For each $i \in \{2, 3, \ldots, n-1\}$, $w_i = c_{i-1}^{-1} v_{k+i} c_i$.

Relabeling the sequence $(c, c_1, c_2, \ldots, c_{n-1})$ we obtain the desired result. □

Let C_1 and C_2 be subgroups of a group G. An equivalence relation can be defined on G as follows: For each pair of elements x and y of G, x is related to y if there exist $c_1 \in C_1$ and $c_2 \in C_2$ such that $x = c_1 y c_2$. A *double coset of G mod C_1 on the left and C_2 on the right* (or briefly a *double coset of G*) is an equivalence class of this equivalence relation. If $x \in G$, then the equivalence class containing x is denoted by $C_1 x C_2$.

Let $(w_1, w_2, \ldots, w_n)$ be a reduced sequence of the free product with amalgamation $G *_C H$. The *sequence of double cosets associated with* $(w_1, w_2, \ldots, w_n)$ is the sequence $(C w_1 w_2 C, C w_1 w_2 C, \ldots, C w_{n-1} w_n C, C w_n w_1 C)$.

Using the above notation, the next corollary (which to our knowledge provides a new invariant of conjugacy classes) follows directly from Theorem 2.8.

Corollary 2.9 *Let $n \geq 2$ and let $(w_1, w_2, \ldots, w_n)$ and $(v_1, v_2, \ldots, v_n)$ be cyclically reduced sequences such that the products $w_1 w_2, \cdots w_n$ and $v_1 v_2 \cdots v_n$ are conjugate. Then the sequence of double cosets associated with $(w_1, w_2, \ldots, w_n)$ is a cyclic permutation of the sequence of double cosets associated with $(v_1, v_2, \ldots, v_n)$.*

Definition 2.10 Let C be a subgroup of a group G and let g be an element of G. Denote by C^g the subgroup of G defined by $g^{-1}Cg$. We say that *C is malnormal in G* if $C^g \cap C = \{1\}$ for every $g \in G \setminus C$.

Lemma 2.11 *Let $G *_C H$ be an free product with amalgamation such that the amalgamating group C is malnormal in G and is malnormal in H. Let a and b be elements of C. Let w_1, w_2 and v_1, v_2 be two reduced sequences such that the sets of double cosets*

$$\{Cw_1aw_2C, Cv_1v_2C\} \text{ and } \{Cw_1w_2C, Cv_1bv_2C\}$$

are equal. Then a and b are conjugate in C. Moreover, if $a \neq 1$ or $b \neq 1$ then v_1 and w_1 are both in G or both in H.

Proof We claim that if $Cw_1aw_2C = Cw_1w_2C$ then $a = 1$. Indeed, if $Cw_1aw_2C = Cw_1w_2C$ then there exist c_1 and c_2 in C such that $c_1w_1aw_2c_2 = w_1w_2$. By Theorem 2.3 there exists $d \in C$ such that

$$w_1 = c_1w_1ad \tag{2.1}$$
$$w_2 = d^{-1}w_2c_2 \tag{2.2}$$

Since C is malnormal in H, by Eq. (2.2), $d = 1$. Thus Eq. (2.1) becomes $w_1 = c_1w_1a$. By malnormality, $a = 1$ and the proof of the claim is complete.

Assume first that $Cw_1aw_2C = Cw_1w_2C$ and $Cv_1v_2C = Cv_1bv_2C$ then, by the claim, $a = 1$ and $b = 1$. Thus a and b are conjugate in C and the result holds in this case.

Now assume that $Cw_1aw_2C = Cv_1bv_2C$ and $Cv_1v_2C = Cw_1w_2C$. By definition of double cosets, there exist $c_1, c_2, c_3, c_4 \in C$ such that $v_1bv_2 = c_1w_1aw_2c_2$ and $v_1v_2 = c_3w_1w_2c_4$. By Theorem 2.3, there exist $d_1, d_2 \in C$ such that

$$v_1b = c_1w_1ad_1 \tag{2.3}$$
$$v_2 = d_1^{-1}w_2c_2 \tag{2.4}$$
$$v_1 = c_3w_1d_2 \tag{2.5}$$
$$v_2 = d_2^{-1}w_2c_4 \tag{2.6}$$

By Eqs. (2.4) and (2.6) and malnormality, $d_1 = d_2$. By Eqs. (2.3) and (2.5) and malnormality, $d_1 = ad_1b^{-1}$. Thus a and b are conjugate in C. Finally by Eq. (2.3), since C is a subgroup of G and a subgroup of H, w_1 and v_1 are both in G or both in H. □

Now we will show that certain pairs of conjugacy classes are distinct by proving that the associated sequences of double cosets of some of their respective representatives are distinct. Warren Dicks suggested the idea of this proof. Our initial proof [4] applied repeatedly the equations given by Theorem 2.8 to derive a contradiction.

Theorem 2.12 *Let $G *_C H$ be a free product with amalgamation. Let i and j be distinct elements of $\{1, 2, \ldots, n\}$ and let a and b be elements of C. Assume all the following:*

(1) *The subgroup C is malnormal in G and is malnormal in H.*
(2) *If i and j have the same parity then a and b are not conjugate in C.*
(3) *Either $a \neq 1$ or $b \neq 1$.*

Then for every cyclically reduced sequence $(w_1, w_2, \ldots w_n)$, the products

$$w_1w_2\cdots w_iaw_{i+1}\cdots w_n \quad \text{and} \quad w_1w_2\cdots w_jbw_{j+1}\cdots w_n$$

are not conjugate.

Proof Let $(w_1, w_2, \dots, w_n)$ be a cyclically reduced sequence. Assume that there exist a and b in C and $i, j \in \{1, 2, \dots, n\}$ as in the hypothesis of the theorem such that the products $w_1 w_2 \cdots w_i a w_{i+1} \cdots w_n$ and $w_1 w_2 \cdots w_j b w_{j+1} \cdots w_n$ are conjugate.

By Corollary 2.9, the sequences of double cosets mod C on the right and the left associated with $(w_1, w_2, \dots, a w_{i+1}, \dots, w_n)$ and with $(w_1, w_2, \dots, b w_{j+1}, \dots, w_n)$ are related by a cyclic permutation. Removing from both sequences the terms denoted by equal expressions yields

$$\{C w_i a w_{i+1} C, C w_j w_{j+1} C\} = \{C w_j b w_{j+1} C, C w_i w_{i+1} C\}.$$

By Lemma 2.11, a and b are conjugate in C. Moreover, w_i and w_j are both in G or both in H. By Remark 2.6, i and j have the same parity, which contradicts hypothesis (2). □

Remark 2.13 It is not hard to construct an example that shows that hypothesis (1) of Theorem 2.12 is necessary. Indeed take, for instance, G and H two infinite cyclic groups generated by x and y respectively. Let C be an infinite cyclic subgroup generated by z. Define $\varphi\colon C \longrightarrow G$ and $\psi\colon C \longrightarrow H$ by $\varphi(z) = x^2$ and $\psi(z) = y^3$. The sequence (x, y) is cyclically reduced. Let $a = x^2$ and $b = x^2$. The products xay and xyb are conjugate.

The following example shows that hypothesis (2) of Theorem 2.12 is necessary. Let $G *_C H$ be a free product with amalgamation, let c be an element of C and let (w_1, w_2) be a reduced sequence. Thus (w_1, w_2, w_1, w_2) is a cyclically reduced sequence. On the other hand, the products $w_1 c w_2 w_1 w_2$ and $w_1 w_2 w_1 c w_2$ are conjugate.

The next result states certain elements of an amalgamated free product are not conjugate.

Theorem 2.14 *Let $G *_C H$ be a free product with amalgamation. Let a and b be elements of C and let and $i, j \in \{1, 2, \dots, n\}$. Assume that for every $g \in (G \cup H) \setminus C$, g and g^{-1} are not in the same double coset relative to C, i.e., $g^{-1} \notin CgC$. Then for every cyclically reduced sequence $(w_1, w_2, \dots w_n)$ the products*

$$w_1 w_2 \cdots w_i a w_{i+1} \cdots w_n \quad \textit{and} \quad w_n^{-1} w_{n-1}^{-1} \cdots w_{j+1}^{-1} b w_j^{-1} \cdots w_1^{-1}$$

are not conjugate.

Proof Assume that the two products are conjugate. Observe that the sequences

$$(w_1, w_2, \dots, w_i a, w_{i+1}, \dots, w_n) \quad \text{and} \quad (w_n^{-1}, w_{n-1}^{-1}, \dots, w_{j+1}^{-1} b, w_j^{-1}, \dots, w_1^{-1})$$

are cyclically reduced. By Theorem 2.8 there exists an integer k such that for every $h \in \{1, 2, \dots, n\}$, w_h and w_{1-h+k}^{-1} are both in G or both in H. Then w_h and w_{1-h+k} are both in G or both in H. By Remark 2.6, n is even and h and $1 - h + k$ have the same parity. Thus k is odd.

Set $l = \frac{k+1}{2}$. By Theorem 2.8, w_l and $w_{1-l+k}^{-1} = w_l^{-1}$ are in the same double coset of G, (if $l \in \{i, j - k\}$, either w_l or w_{1-l+k} may appear multiplied by a or b in the equations of Theorem 2.8 but this does not change the double coset.) Thus w_l and w_l^{-1} are in the same double coset mod C of G, contradicting our hypothesis. □

3 Oriented separating simple loops

The goal of this section is to prove Theorem 3.4, which gives a way to compute the bracket of a separating simple closed curve x and the product of the terms of a cyclically reduced

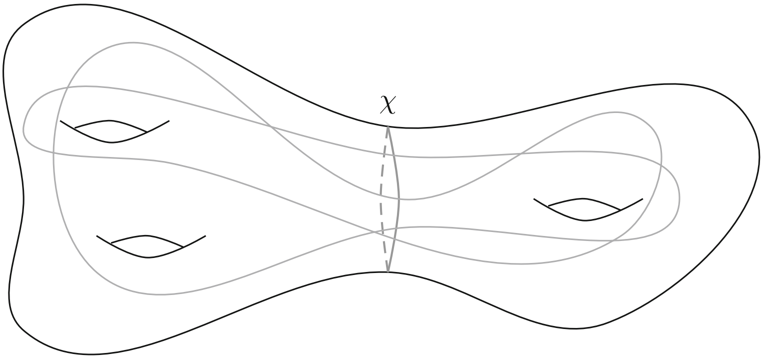

Fig. 1 A separating curve χ intersecting another curve

sequence given by the amalgamated free product determined by x (see Fig. 1). This theorem is proved by finding in Lemma 3.2 appropriate representatives for the separating simple closed curve x and the terms of the cyclically reduced sequence.

Through the rest of these pages, a *surface* will mean a connected oriented surface. We denote such a surface by Σ. The fundamental group of Σ will be denoted by $\pi_1(\Sigma, p)$ where $p \in \Sigma$ is the basepoint or by $\pi_1(\Sigma)$ when the basepoint does not play a role in the discussion. By a *curve* we will mean a closed oriented curve on Σ. We will use the same letter to denote a curve and its image in Σ.

Let χ be a separating non-trivial simple curve on Σ, non-parallel to a boundary component of Σ. Choose a point $p \in \chi$ to be the basepoint of each of the fundamental groups which will appear in this context. Denote by Σ_1 the union of χ and one of the connected components of $\Sigma \backslash \chi$ and by Σ_2 the union of χ with the other connected component.

Remark 3.1 As a consequence of van Kampen's theorem (see [16]) $\pi_1(\Sigma, p)$ is canonically isomorphic to the free product of $\pi_1(\Sigma_1, p)$ and $\pi_1(\Sigma_2, p)$ amalgamating the subgroup $\pi_1(\chi, p)$, where the monomorphisms $\pi_1(\chi, p) \longrightarrow \pi_1(\Sigma_1, p)$ and $\pi_1(\chi, p) \longrightarrow \pi_1(\Sigma_2, p)$ are the induced by the respective inclusions.

Lemma 3.2 *Let χ be a separating simple closed curve on Σ. Let $(w_1, w_2, \ldots, w_n)$ be a cyclically reduced sequence for the amalgamated product of Remark 3.1. Then there exists a sequence of curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that for each $i \in \{1, 2, \ldots, n\}$ then each of the following holds.*

(1) *The curve γ_i is a representative of w_i.*
(2) *The curve γ_i is alternately in Σ_1 and Σ_2, that is, either $\gamma_i \subset \Sigma_j$ where $i \equiv j \pmod 2$ for all i and $j \in \{1, 2\}$, or $\gamma_i \subset \Sigma_j$ where $i \equiv j + 1 \pmod 2$ for all i and $j \in \{1, 2\}$.*
(3) *The point p is the basepoint of γ_i.*
(4) *The point p is not a self-intersection point of γ_i. In other words, γ_i passes through p exactly once.*

Moreover, the product $\gamma_1\gamma_2 \cdots \gamma_n$ is a representative of the product $w_1w_2 \cdots w_n$ and the curve $\gamma_1\gamma_2 \cdots \gamma_n$ intersects χ transversally with multiplicity n.

Proof For each $i \in \{1, 2, \ldots, n\}$, take a representative γ_i in of w_i. Notice that $\gamma_i \subset \Sigma_1$ or $\gamma_i \subset \Sigma_2$. We homotope γ_i if necessary, in such a way that γ_i passes through p only once, at the basepoint p (Fig. 2).

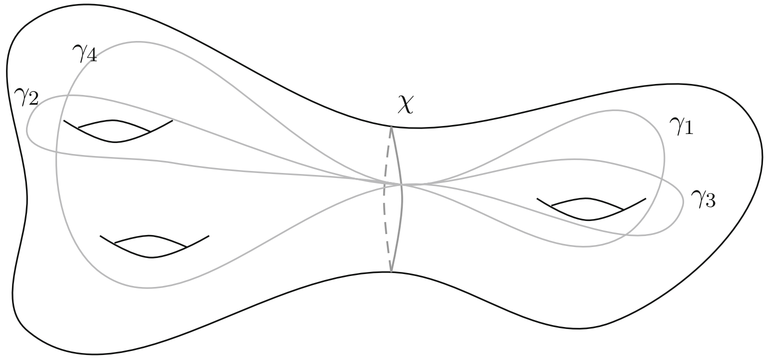

Fig. 2 The representative of Lemma 3.2

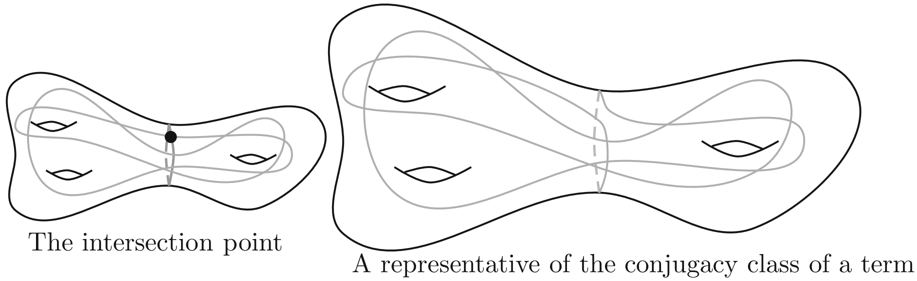

Fig. 3 An intersection point (*left*) and the corresponding term of the bracket (*right*)

Since each γ_i intersects χ only at p, the product $\gamma_1\gamma_2\cdots\gamma_n$ intersects χ exactly n times. Each of these intersections happens when the curve $\gamma_1\gamma_2\cdots\gamma_n$ passes from Σ_1 to Σ_2 or from Σ_2 to Σ_1. This implies that these n intersection points of χ with $\gamma_1\gamma_2\cdots\gamma_n$ are transversal. □

Let Σ be an oriented surface. The Goldman bracket [13] is a Lie bracket defined on the vector space generated by all free homotopy classes of oriented curves on the surface Σ. We recall the definition: For each pair of homotopy classes a and b, consider representatives, also called a and b, that only intersect in transversal double points. The bracket of $[a, b]$ is defined as the signed sum over all intersection points P of a and b of free homotopy class of the curve that goes around a starting and ending at P and then goes around b starting and ending at P. The sign of the term at an intersection point P is the intersection number of a and b at P. (See Fig. 3.)

The above definition can be extended to consider pairs of representatives where branches intersect transversally but triple (and higher) points are allowed. Indeed, take a pair of such representatives a and b. The Goldman Lie bracket is the sum over the intersection of pairs of small arcs, of the conjugacy classes of the curve obtained by starting in an intersection point and going along the a starting in the direction of the first arc, and then going around b starting in the direction of the second arc. The sign is determined by the pair of tangents of the ordered oriented arcs at the intersection point (Fig. 3).

Notation 3.3 The Goldman bracket $[a, b]$ is computed for pairs of conjugacy classes a and b of curves on Σ. In order to make the notation lighter, we will abuse notation by writing

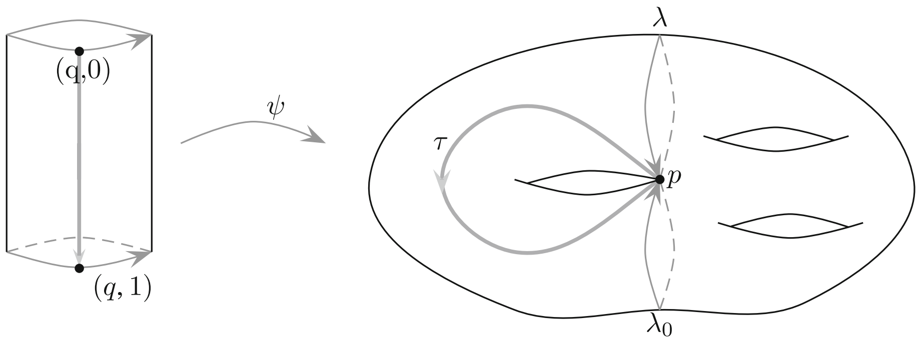

Fig. 4 The map ψ of Lemma 5.1

$[u, v]$, where u, v are elements of the fundamental group of Σ. By $[u, v]$, then, we will mean the bracket of the conjugacy class of u and the conjugacy class of v.

Theorem 3.4 *Let x be a conjugacy class of $\pi_1(\Sigma, p)$ which can be represented by a separating simple closed curve χ. Let $(w_1, w_2, \ldots, w_n)$ be a cyclically reduced sequence for the amalgamated product determined by χ in Remark 3.1. Then $[w_1, x] = 0$. Moreover, if $n > 1$ then there exists $s \in \{1, -1\}$ such that the bracket is given by*

$$s[w_1 w_2 \cdots w_n, x] = \sum_{i=1}^{n} (-1)^i w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n.$$

Proof For each curve γ there exists a representative of the null-homotopic class disjoint from γ so the result holds when $n = 0$. We prove now that $[w_1, x] = 0$. By definition of a cyclically reduced sequence, $w_1 \in \pi_1(\Sigma_1, p)$ or $w_1 \in \pi_1(\Sigma_2, p)$. Suppose that $w_1 \in \pi_1(\Sigma_1, p)$ (the other possibility is analogous.) Choose a curve $\gamma_1 \subset \Sigma_1$ which is a representative of w_1. Since $\gamma_1 \subset \Sigma_1$, we can homotope γ_1 to a curve which has no intersection with χ. Clearly this is a free homotopy which does not fix the basepoint p. This shows that w_1 and x have disjoint representatives, and then $[w_1, x] = 0$.

Now assume that $n > 1$. Let $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ be the sequence given by Lemma 3.2 for $(w_1, w_2, \ldots, w_n)$.

The loop product $\gamma_1\gamma_2 \cdots \gamma_n$ is a representative of the group product $w_1 w_2 \cdots w_n$.

Every intersection of $\gamma_1\gamma_2 \cdots \gamma_n$ with χ occurs when $\gamma_1\gamma_2 \cdots \gamma_n$ leaves one connected component of $\Sigma \setminus \chi$ to enter the other connected component. This occurs between each γ_i and γ_{i+1}. (Recall we are using Notation 2.5 so the intersection between γ_n and γ_1 is considered.)

For each $i \in \{1, 2, \ldots, n\}$ denote by p_i the intersection point of χ and $\gamma_1\gamma_2 \cdots \gamma_n$ between γ_i and γ_{i+1}. The loop product $\gamma_1\gamma_2 \cdot \gamma_i \chi \gamma_{i+1} \cdots \gamma_n$ is a representative of the conjugacy class of the term of the Goldman Lie bracket corresponding to p_i. Thus for each $i \in \{1, 2, \ldots, n\}$, the conjugacy class of the term of the bracket corresponding to the intersection point p_i has $w_1 \cdots w_i x w_{i+1} \cdots w_n$ as representative.

Let $i, j \in \{1, 2, \ldots, n\}$ with different parity. Assume that $w_i \in \pi_1(\Sigma_1, p)$, (the case $w_i \in \pi_2(\Sigma_1, p)$ is similar.) The tangent vector of $\gamma_1\gamma_2 \cdots \gamma_n$ at p_i points towards Σ_2 and the tangent vector of $\gamma_1\gamma_2 \cdots \gamma_n$ at p_j points towards Σ_1. This shows that the signs of the bracket terms corresponding to p_i and p_j are opposite, completing the proof. □

Remark 3.5 In Theorem 3.4, all the intersections of the chosen representatives of $w_1 w_2 \cdots w_n$ and x occur at the basepoint p. The representative ω of $w_1 w_2 \cdots w_n$ intersects x in a point which is a multiple self-intersection point of ω. This does not present any difficulty in computing the bracket, because the intersection points of both curves are still transversal double points.

4 HNN extensions

This section is the HNN counterpart of Sect. 2 and the statements, arguments and posterior use of the statements are similar. The main goal consists in proving that the products of certain cyclically reduced sequences cannot be conjugate (Theorems 4.16 and 4.19.) This result will be used to show that the pairs of terms of the bracket of certain conjugacy classes with opposite sign do not cancel.

Let G be a group, let A and B be two subgroups of G and let $\varphi\colon A \longrightarrow B$ be an isomorphism. Then the *HNN extension of G relative to A, B and φ with stable letter t (or, more briefly, the HNN extension of G relative to φ)* will be denoted by $G^{*\varphi}$ and is the group obtained by taking the quotient of the free product of G and the free group generated by t by the normal subgroup generated by $t^{-1}at\varphi(a)^{-1}$ for all $a \in A$. (See [25] for detailed definitions.)

Definition 4.1 Consider an HNN extension $G^{*\varphi}$. Let n be a non-negative integer and for each $i \in \{1, 2, \ldots, n\}$, let $\varepsilon_i \in \{1, -1\}$ and g_i be an element of G. A finite sequence $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n}, g_n)$ is said to be *reduced* if there is no consecutive subsequence of the form (t^{-1}, g_i, t) with $g_i \in A$ or (t, g_j, t^{-1}) with $g_j \in B$.

The following result is the analogue of Theorem 2.2 for HNN extensions (see [25] or [8].)

Theorem 4.2 (1) (*Britton's lemma*) *If the sequence $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n}, g_n)$ is reduced and $n \geq 1$ then the product $g_0t^{\varepsilon_1}g_1t^{\varepsilon_2} \cdots g_{n-1}t^{\varepsilon_n}g_n$ is not the identity in the HNN extension $G^{*\varphi}$.* (2) *Every element g of $G^{*\varphi}$ can be written as a product $g_0t^{\varepsilon_1}g_1t^{\varepsilon_2} \cdots g_{n-1}t^{\varepsilon_n}g_n$ where the sequence $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n}, g_n)$ is reduced.*

As in the case of Theorem 2.3, we include the proof of the next known result because we were unable to find it in the literature.

Theorem 4.3 *Suppose that the equality*

$$g_0t^{\varepsilon_1}g_1t^{\varepsilon_2} \cdots g_{n-1}t^{\varepsilon_n}g_n = h_0t^{\eta_1}h_1t^{\eta_2} \cdots h_{n-1}t^{\eta_n}h_n$$

holds where $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n}, g_n)$ and $(h_0, t^{\eta_1}, h_1, t^{\eta_2} \ldots, h_{n-1}, t^{\eta_n}, h_n)$ are reduced sequences.

Then for each $i \in \{1, 2, \ldots, n\}$, $\varepsilon_i = \eta_i$. Moreover, there exists a sequence of elements $(c_1, c_2, \ldots, c_n)$ in $A \cup B$ such that

(1) $g_0 = h_0c_1$
(2) $g_n = \varphi^{\varepsilon_n}(c_n^{-1})h_n$
(3) *For each $i \in \{1, 2, \ldots, n-1\}$, $g_i = \varphi^{\varepsilon_i}(c_i^{-1})h_ic_{i+1}$*
(4) *For each $i \in \{1, 2, \ldots, n\}$, $c_i \in A$ if $\varepsilon_i = 1$ and $c_i \in B$ if $\varepsilon_i = -1$.*

Proof If the two products are equal then

$$h_n^{-1}t^{-\eta_n} \cdots t^{-\eta_2}h_1^{-1}t^{-\eta_1}h_0^{-1}g_0t^{\varepsilon_1}g_1t^{\varepsilon_2} \cdots g_{n-1}t^{\varepsilon_n}g_n = 1. \tag{4.1}$$

By Theorem 4.2(1) the sequence that yields the product on the left hand side of Eq. (4.1) is not reduced. This implies that $\varepsilon_1 = \eta_1$. Moreover, if $\varepsilon_1 = 1$ then $h_0^{-1}g_0 \in A$ and if $\varepsilon_1 = -1$ then $h_0^{-1}g_0 \in B$.

Denote the product $h_0^{-1} g_0$ by c_1. Thus $g_0 = h_0 c_1$. By definition of HNN extension, we can replace $t^{-\varepsilon_1} c_1 t^{\varepsilon_1}$ by $\varphi^{\varepsilon_1}(c_1)$ in Eq. (4.1) to obtain,

$$h_n^{-1} t^{-\eta_n} \cdots t^{-\eta_2} h_1^{-1} \varphi^{\varepsilon_1}(c_1) g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n} g_n = 1 \tag{4.2}$$

By Theorem 4.2, the sequence yielding the product of the left hand side of Eq. (4.2) is not reduced. Hence $\varepsilon_2 = \eta_2$ and if $\varepsilon_2 = 1$ then $h_1^{-1} \varphi^{\varepsilon_1}(c_1) g_1 \in A$ and if $\varepsilon_2 = -1$ then $h_1^{-1} \varphi^{\varepsilon_1}(c_1) g_1 \in B$.

Denote by c_2 the product $h_1^{-1} \varphi^{\varepsilon_1}(c_1) g_1$. Thus $g_1 = \varphi^{\varepsilon_1}(c_1^{-1}) h_1 c_2$.

By applying these arguments, we can complete the proof by induction. □

Definition 4.4 Let n be a non-negative integer. A sequence of elements of $G*^{\varphi}$, $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$ is said to be *cyclically reduced* if all its cyclic permutations of are reduced.

We could not find a direct proof in the literature of the first statement of our next so we include it here. (The second statement also follows from Theorem 4.7 but it is a direct consequence of our proof.)

Theorem 4.5 *Let s be a conjugacy class of $G*^{\varphi}$. Then there exists a cyclically reduced sequence such that the product of its terms is a representative of s. Moreover, every cyclically reduced sequence with product in s has the same number of terms.*

Proof If s has a representative in G, the result follows directly. So we can assume that s has no representatives in G.

By Theorem 4.2(2), the set of reduced sequences with product in s is not empty. Thus it is possible to choose, among all such sequences, one that makes the number of terms the smallest possible. Let $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{m-1}, t^{\varepsilon_m})$ be such a sequence. We claim that $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{m-1}, t^{\varepsilon_m})$ is cyclically reduced.

Indeed, if $m \in \{0, 1\}$, the sequence has the form (g_0) or (g_0, t^{ε_1}) and so it is cyclically reduced. Assume now that $m > 1$.

If $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{m-1}, t^{\varepsilon_m})$ is not cyclically reduced then one of the following statements holds:

(1) $\varepsilon_1 = 1$, $\varepsilon_m = -1$ and $g_0 \in A$.
(2) $\varepsilon_1 = -1$, $\varepsilon_m = 1$ and $g_0 \in B$.

We prove the result in case (1). (Case (2) can be treated with similar ideas.) In this case, $t^{-\varepsilon_m} g_0 t^{\varepsilon_1} = \varphi(g_0) \in B$.

The sequence $(g_{m-1}\varphi(g_0)g_1, t^{\varepsilon_2}, \ldots, g_{m-2}, t^{\varepsilon_{m-1}})$ is reduced, has product in s and has strictly fewer terms than the sequence $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{m-1}, t^{\varepsilon_m})$. This contradicts our assumption that our original sequence has the smallest number of terms. Thus our proof is complete. □

Notation 4.6 By definition, the cyclically reduced sequences of a given HNN extension have the form $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$. From now on, we will make use of the following convention: For every integer h, g_h will denote g_i where i is the unique integer in $\{0, 1, 2, \ldots, n-1\}$ such that n divides $i - h$. Analogously, ε_h will denote ε_i where i is the unique integer in $\{1, 2, \ldots, n\}$ such that n divides $i - h$.

The next result is due to Collins and gives necessary conditions for two cyclically reduced sequence have conjugate product (see [25].)

Theorem 4.7 (*Collins′Lemma*) *Let $n \geq 1$ and let $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$ and $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \ldots, h_{m-1}, t^{\eta_m})$ be two cyclically reduced sequences such that their products are conjugate. Then $n = m$ and there exist $c \in A \cup B$ and $k \in \{0, 1, 2, \ldots, n-1\}$ such that the following holds:*

(1) $\eta_k = \varepsilon_n$,
(2) $c \in A$ *if* $\varepsilon_n = -1$ *and* $c \in B$ *if* $\varepsilon_n = 1$,
(3) $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n} = c^{-1} h_k t^{\eta_{k+1}} h_{k+1} t^{\eta_{k+2}} \cdots h_{k+n-1} t^{\eta_k} c.$

By Theorems 4.3 and 4.7 and arguments like those of Theorem 2.8, we obtain the following result.

Theorem 4.8 *Let n be a positive integer and let $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$ and $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \ldots, h_{n-1}, t^{\eta_n})$ be cyclically reduced sequences such that the products*

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n} \quad \text{and} \quad h_0 t^{\eta_1} h_1 t^{\eta_2} \cdots h_{n-1} t^{\eta_n}$$

are conjugate. Then there exists an integer k such that for each $i \in \{1, 2, \ldots, n\}$, $\varepsilon_i = \eta_{i+k}$. Moreover, there exists a sequence of elements $(c_1, c_2, \ldots, c_n)$ in $A \cup B$ such that for each $i \in \{1, 2, \ldots, n\}$, $c_i \in A$ if $\varepsilon_i = 1$ and $c_i \in B$ if $\varepsilon_i = -1$ and

$$g_i = \varphi^{\eta i+k}(c_{i+k}^{-1}) h_{i+k} c_{i+k+1}$$

Notation 4.9 Let $G^{*\varphi}$ be an HNN extension, where $\varphi\colon A \longrightarrow B$. We denote the subgroup A by C_1 and the subgroup B by C_{-1}.

A direct consequence of Theorem 4.8 follows.

Corollary 4.10 *Let $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n})$ and $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \ldots, h_{n-1}, t^{\eta_n})$ be cyclically reduced sequences such that the products*

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n} \quad \text{and} \quad h_0 t^{\eta_1} h_1 t^{\eta_2} \cdots h_{n-1} t^{\eta_n}$$

are conjugate. If $n \geq 1$ then there exists an integer k such that for each $i \in \{1, 2, \ldots, n\}$, $\varepsilon_i = \eta_{i+k}$ and g_i belongs to the double coset $C_{-\varepsilon_i} h_{i+k} C_{\varepsilon_{i+1}}$.

Remark 4.11 If $\varepsilon \in \{1, -1\}$ and $a \in C_\varepsilon$ then $at^\varepsilon = t^\varepsilon \varphi^\varepsilon(a)$ in $G^{*\varphi}$.

Definition 4.12 Let $n \geq 2$ and let $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$ be a cyclically reduced sequence. The *sequence of double cosets associated with* $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$ is the sequence of double cosets

$$(C_{\varepsilon_i} t^{\varepsilon_i} g_i t^{\varepsilon_{i+1}} g_{i+1} t^{\varepsilon_{i+2}} C_{-\varepsilon_{i+2}})_{0 \leq i \leq n-1}.$$

Lemma 4.13 *Let $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n})$ and $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \ldots h_{n-1}, t^{\eta_n})$ be two cyclically reduced sequences whose products are conjugate and such that $n \geq 2$. Then the sequence of double cosets associated with $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n})$ is a cyclic permutation of the sequence of double cosets associated with $(h_0, t^{\eta_1}, h_1 t^{\eta_2}, \ldots h_{n-1}, t^{\eta_n})$.*

Proof Let $k \in \{1, 2, \ldots, n\}$ and $(c_1, c_2, \ldots, c_n)$ be a finite sequence of elements in $A \cup B$ given by Theorem 4.8 for the sequences $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n})$ and $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \ldots h_{n-1}, t^{\eta_n})$. By cyclically rotating $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \ldots h_{n-1}, t^{\eta_n})$ if necessary, we can assume that $k = 0$.

Let $i \in \{1, 2, \ldots, n\}$. We will complete this proof by showing that the i-th double coset of the sequence $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n})$ equals the i-th double coset of the sequence of $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \ldots h_{n-1}, t^{\eta_n})$.

By Theorem 4.8, $\varepsilon_j = \eta_j$ for each $j \in \{1, 2, \ldots, n\}$ and

$$g_i t^{\varepsilon_{i+1}} g_{i+1} = \varphi^{\eta_i}(c_i^{-1}) h_i c_{i+1} t^{\eta_{i+1}} \varphi^{\eta_{i+1}}(c_{i+1}^{-1}) h_{i+1} c_{i+2}$$

By Remark 4.11, $c_{i+1} t^{\eta_{i+1}} \varphi^{\eta_{i+1}}(c_{i+1}^{-1}) = t^{\eta_{i+1}}$. Thus

$$g_i t^{\varepsilon_{i+1}} g_{i+1} = \varphi^{\eta_i}(c_i^{-1}) h_i t^{\eta_{i+1}} h_{i+1} c_{i+2}$$

By Theorem 4.8, $c_i \in C_{\varepsilon_i}$ and $c_{i+2} \in C_{\varepsilon_{i+2}}$. Therefore $\varphi^{\eta_i}(c_i^{-1}) \in C_{-\varepsilon_i}$. Thus by Remark 4.11

$$t^{\varepsilon_i} g_i t^{\varepsilon_{i+1}} g_{i+1} t^{\varepsilon_{i+2}} \in t^{\varepsilon_i} C_{-\varepsilon_i} h_i t^{\eta_{i+1}} h_{i+1} C_{\varepsilon_{i+2}} t^{\varepsilon_{i+2}}$$

Consequently, by Remark 4.11, $t^{\varepsilon_i} g_i t^{\varepsilon_{i+1}} g_{i+1} t^{\varepsilon_{i+2}}$ is in the i-th double coset associated with $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \ldots h_{n-1}, t^{\eta_n})$ and our proof is complete. □

Definition 4.14 Let $G^{*\varphi}$ be an HNN extension, where $\varphi\colon A \longrightarrow B$. We say that $G^{*\varphi}$ is *separated* if $A \cap B^g = \{1\}$ for all $g \in G$.

Lemma 4.15 *Let $(t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, g_2, t^{\varepsilon_3})$ and $(t^{\eta_1}, h_1, t^{\eta_2}, h_2, t^{\eta_3})$ be two reduced sequences. Let $a \in C_{\varepsilon_2}$ and let $b \in C_{\eta_2}$. Suppose that $G^{*\varphi}$ is separated and that A and B are malnormal in $G^{*\varphi}$. Then the following statements hold.*

(1) If the double cosets $C_{\varepsilon_1} t^{\varepsilon_1} g_1 a t^{\varepsilon_2} g_2 t^{\varepsilon_3} C_{-\varepsilon_3}$ and $C_{\varepsilon_1} t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 t^{\varepsilon_3} C_{-\varepsilon_3}$ are equal, then $a = 1$.

(2) If the following sets of double cosets $\{C_{\varepsilon_1} t^{\varepsilon_1} g_1 a t^{\varepsilon_2} g_2 t^{\varepsilon_3} C_{-\varepsilon_3}, C_{\eta_1} t^{\eta_1} h_1 t^{\eta_2} h_2 t^{\eta_3} C_{\varepsilon_3}\}$ and $\{C_{\varepsilon_1} t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 t^{\varepsilon_3} C_{-\varepsilon_3}, C_{\eta_1} t^{\eta_1} h_1 b t^{\eta_2} h_2 t^{\eta_3} C_{\varepsilon_3}\}$ are equal then a and b are conjugate by an element of $A \cup B$. Moreover, if $a \neq 1$ or $b \neq 1$, then $\varepsilon_2 = \eta_2$.

Proof We first prove (1). By Remark 4.11,

$$t^{\varepsilon_1} C_{-\varepsilon_1} g_1 a t^{\varepsilon_2} g_2 C_{\varepsilon_3} t^{\varepsilon_3} = t^{\varepsilon_1} C_{-\varepsilon_1} g_1 t^{\varepsilon_2} g_2 C_{\varepsilon_3} t^{\varepsilon_3}.$$

Cross out t^{ε_1} and t^{ε_3} at both sides of the above equation. Then there exist $d_1 \in C_{\varepsilon_1}$ and $d_2 \in C_{\varepsilon_3}$ such that $g_0 t^{\varepsilon_2} g_1 = d_1 g_0 a t^{\varepsilon_2} g_1 d_2$. By Theorem 4.3, there exists $c \in C_{\varepsilon_2}$ such that

$$g_0 = d_1 g_0 a c \tag{4.3}$$

$$g_1 = \varphi^{\varepsilon_2}(c^{-1}) g_1 d_2. \tag{4.4}$$

By malnormality, separability, and Eq. (4.4), $\varphi^{\varepsilon_2}(c^{-1}) = 1$ and so $c = 1$. By malnormality, separability and Eq. (4.3), $ac = 1$. Hence $a = 1$.

Now we prove (2). By statement (1), the first (resp. second) element listed in the first set is equal to the second (resp. first) element listed on the second set. By Theorem 4.3, for each $i \in \{1, 2, 3\}$, $\varepsilon_i = \eta_i$. By arguing as in the proof of statement (1), we can deduce that there exist c_1 and d_1 in $C_{-\varepsilon_1}$ and c_3 and d_3 in C_{ε_3}, such that

$$g_1 a t^{\varepsilon_2} g_2 = c_1 h_1 b t^{\varepsilon_2} h_2 c_3 \quad \text{and} \quad g_1 t^{\varepsilon_2} g_2 = d_1 h_1 t^{\varepsilon_2} h_2 d_3$$

By Theorem 4.3, there exists a pair of elements, x and y in C_{ε_2} such that

$$g_0a = c_1h_0bx \tag{4.5}$$
$$g_1 = \varphi^{\varepsilon_2}(x^{-1})h_1c_3. \tag{4.6}$$
$$g_0 = d_1h_0y \tag{4.7}$$
$$g_1 = \varphi^{\varepsilon_2}(y^{-1})h_1d_3. \tag{4.8}$$

Since φ is an isomorphism, by malnormality and separability and Eqs. (4.6) and (4.8), $x = y$. Analogously, by malnormality and separability and Eqs. (4.5) and (4.7), $bxa^{-1} = y$. Therefore a and b are conjugate by x. Since $x \in A \cup B$, the proof is complete. □

The next theorem gives necessary conditions for certain products of cyclically reduced sequences to not be conjugate.

Theorem 4.16 *Let $G^{*\varphi}$ be an HNN extension. Let $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n})$ be a cyclically reduced sequence. and let i and j be elements of $\{1, 2, \ldots, n\}$. Let a be an element of C_{ε_i} and let b be an element of C_{ε_j}. Moreover, assume that the following conditions hold:*

(1) The subgroups A and B are malnormal in G.
(2) The HNN extension $G^{*\varphi}$ is separated.
(3) If $\varepsilon_i = \varepsilon_j$ then a and b are not conjugate by an element of $A \cup B$.
(4) Either $a \neq 1$ or $b \neq 1$.

Then the products

$$g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{i-1}at^{\varepsilon_i}g_i\cdots g_{n-1}t^{\varepsilon_n} \quad \text{and} \quad g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{j-1}bt^{\varepsilon_j}g_j\cdots g_{n-1}t^{\varepsilon_n}$$

are not conjugate.

Proof Assume that the products are conjugate. Without loss of generality, we can also assume that a is not the identity.

If $n = 1$ the proof of the result is direct. Hence we can assume $n \geq 2$. The sequences

$(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_ia, t^{\varepsilon_{i+1}}, \ldots g_{n-1}, t^{\varepsilon_n})$ and $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_jb, t^{\varepsilon_{j+1}}, \ldots g_{n-1}, t^{\varepsilon_n})$

are cyclically reduced. By Lemma 4.13 both sequences are associated with the same cyclic sequence of double cosets. If $i = j$ the result is a consequence of Lemma 4.15(1). Hence $i \neq j$. By Remark 4.11,

$$C_{\varepsilon_{i-2}}t^{\varepsilon_{i-2}}g_{i-2}t^{\varepsilon_{i-1}}g_{i-1}at^{\varepsilon_i}C_{-\varepsilon_i} = C_{\varepsilon_{i-2}}t^{\varepsilon_{i-2}}g_{i-2}t^{\varepsilon_{i-1}}g_{i-1}t^{\varepsilon_i}C_{-\varepsilon_i}$$

and

$$C_{\varepsilon_{j-2}}t^{\varepsilon_{j-2}}g_{j-2}t^{\varepsilon_{j-1}}g_{j-1}bt^{\varepsilon_j}C_{-\varepsilon_j} = C_{\varepsilon_{j-2}}t^{\varepsilon_{j-2}}g_{j-2}t^{\varepsilon_{j-1}}g_{j-1}t^{\varepsilon_j}C_{-\varepsilon_j}.$$

As in the proof of Theorem 2.12, crossing out the elements in the sequences of double cosets with equal expressions shows that the sets below

$$\{C_{\varepsilon_{i-1}}t^{\varepsilon_{i-1}}g_{i-1}at^{\varepsilon_i}g_it^{\varepsilon_{i+1}}C_{-\varepsilon_{i+1}}, C_{\varepsilon_{j-1}}t^{\varepsilon_{j-1}}g_{j-1}t^{\varepsilon_j}g_jt^{\varepsilon_{j+1}}C_{-\varepsilon_{j+1}}\}$$

$$\{C_{\varepsilon_{i-1}}t^{\varepsilon_{i-1}}g_{i-1}t^{\varepsilon_i}g_it^{\varepsilon_{i+1}}C_{-\varepsilon_{i+1}}, C_{\varepsilon_{j-1}}t^{\varepsilon_{j-1}}g_{j-1}bt^{\varepsilon_j}g_jt^{\varepsilon_{j+1}}C_{-\varepsilon_{j+1}}\}$$

are equal. By Lemma 4.15, $\varepsilon_i = \varepsilon_j$ and a and b are conjugate by an element of $A \cup B$. This contradicts our hypothesis and so the proof is complete. □

Remark 4.17 The following example shows that hypotheses (1) or (2) of Theorem 4.16 are necessary. Let G be the direct sum $\mathbb{Z} \oplus \mathbb{Z}$ of $\mathbb{Z}$ and $\mathbb{Z}$. Let $A = \mathbb{Z} \oplus \{0\}$ and let $B = \{0\} \oplus \mathbb{Z}$ considered as subgroups of $\mathbb{Z} \oplus \mathbb{Z}$. Define $\varphi\colon A \longrightarrow B$ by $\varphi(x, 0) = (0, x)$. Let a be an element of A. Since $\mathbb{Z} \oplus \mathbb{Z}$ is commutative,

$$(0, 1)t^{-1}(1, 1)t^{-1}(1, 1)t(0, -1) = t^{-1}(2, 1)t^{-1}(0, 1)t$$

Thus the sequences $t^{-1}, (1, 1), t^{-1}, (0, 1)(1, 0), t$ and $t^{-1}, (1, 1)(1, 0), t^{-1}, (0, 1), t$ have conjugate products. On the other hand, hypotheses (3) and (4) of Theorem 4.16 hold for these sequences.

The following example shows that hypothesis (3) of Theorem 4.16 is necessary. Let G^* be an HNN extension relative to an isomorphism $\varphi\colon A \longrightarrow B$. Let $g \in G \setminus A$ and let $a \in A$. The sequence (g, t, g, t) is cyclically reduced and the product of the sequences (ga, t, g, t) and (g, t, ga, t) are conjugate.

The next auxiliary lemma will be used in the proof of Theorem 4.19. The set of congruence classes modulo n is denoted by $\mathbb{Z}/n\mathbb{Z}$.

Lemma 4.18 *Let n and k be integers. Let $\widehat{F}\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ be induced by the map on the integers given by the formula $F(x) = -x + k$. If n is odd or k is even then $\widehat{F}$ has a fixed point.*

Proof If k is even then $\frac{k}{2}$ is integer and a fixed point of F. Thus $\widehat{F}$ has a fixed point.

On the other hand, the map $\widehat{F}$ has a fixed point whenever the equation $2x \equiv -k \pmod{n}$ has a solution. If n is odd this equation has a solution because 2 has an inverse in $\mathbb{Z}/n\mathbb{Z}$. This completes the proof. □

We will use Notation 4.9 for the statement and proof of the next result.

Theorem 4.19 *Let $G^{*\varphi}$ be an HNN extension; let $(g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \ldots g_{n-1} t^{\varepsilon_n})$ be a cyclically reduced sequence. Let i and j be elements of $\{1, 2, \ldots, n\}$. Let a be an element of C_{ε_i} and let b an element of $C_{-\varepsilon_j}$. Assume that for each $g \in G$, g^{-1} does not belong to the set $(A \cup B)g(A \cup B)$. Then the products*

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{i-1} a t^{\varepsilon_i} g_i \cdots g_{n-1} t^{\varepsilon_n} \text{ and}$$

$$g_{n-1}^{-1} t^{-\varepsilon_{n-1}} g_{n-2}^{-1} t^{-\varepsilon_{n-2}} \cdots g_j^{-1} b t^{-\varepsilon_j} \cdots t^{-\varepsilon_1} g_0^{-1} t^{-\varepsilon_n}$$

are not conjugate.

Proof Here is a sketch of the proof: If the above products are conjugate then the sequence of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is a rotation of the sequence $(-\varepsilon_n, -\varepsilon_{n-1}, \ldots, -\varepsilon_1)$. Since the terms of those sequences are not zero, a term ε_j of the first sequence cannot correspond to a term of the form $-\varepsilon_h$. This gives conditions of the rotation and n. The same rotation also establishes a correspondence between double cosets of g_h's and double cosets of rotated g_h^{-1}'s. Using the fact that the sequences ε_h's and g_h's are "off" by one, we will show that there exists u such that $g_u^{-1} \in (A \cup B)g_u(A \cup B)$.

Now the detailed proof. Consider the sequence $(s_0, t^{\eta_1}, s_1, t^{\eta_2} \ldots, s_{n-1}, t^{\eta_n})$ defined by $\eta_h = -\varepsilon_{n-h}$ and

$$s_h = \begin{cases} g_{-h-1}^{-1} b & \text{if } -h-1 \equiv j \pmod{n}, \\ g_{-h-1}^{-1} & \text{otherwise.} \end{cases} \tag{4.9}$$

for each h. (Recall Notation 4.6.)

Assume that the products of the hypothesis of the theorem are conjugate. Thus the sequences

$$(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \dots, g_{i-1}a, t^{\varepsilon_i}, g_i, \dots g_{n-1}, t^{\varepsilon_n}) \quad \text{and} \quad (s_0, t^{\eta_1}, s_1, t^{\eta_2} \dots s_{n-1}, t^{\eta_n})$$

are cyclically reduced and have conjugate products. Let k be as in Corollary 4.10 for these two products. Hence

$$\varepsilon_h = \eta_{h+k} = -\varepsilon_{n-h-k}, \tag{4.10}$$

and the following holds:

(1) if $h \not\equiv i-1 \pmod{n}$ then g_h belongs to the double coset $C_{-\varepsilon_h} s_{h+k} C_{\varepsilon_{h+1}}$,
(2) $g_{i-1}a$ belongs to the double coset $C_{-\varepsilon_{i-1}} s_{i+k} C_{\varepsilon_i}$.

By hypothesis, $a \in C_{\varepsilon_i}$. Thus for all h we have

$$g_h \in C_{-\varepsilon_h} s_{h+k} C_{\varepsilon_{h+1}} \tag{4.11}$$

Since for every integer h, $\varepsilon_h \neq 0$, $\varepsilon_h \neq -\varepsilon_h$. Therefore by Eq. (4.10), the map $\widehat{F}$ defined on the integers mod n induced by $F(h) = -h - k$ cannot have fixed points. By Lemma 4.18, n is even and k is odd.

Then $(-k-1)$ is even. By Lemma 4.18, the map $G(h) = -h + (-k-1)$ has a fixed point. Denote this fixed point by u. Thus

$$u + k \equiv -u - 1 \pmod{n} \tag{4.12}$$

Assume that $u \not\equiv j \pmod{n}$. By Eqs. (4.12) and (4.9), $s_{u+k} = s_{-u-1} = g_u^{-1}$. By Eq. (4.11),

$$g_u \in C_{-\varepsilon_u} g_u^{-1} C_{\varepsilon_{u+1}} \subset (A \cup B) g_j (A \cup B),$$

contradicting our hypothesis. Therefore $u \equiv j \pmod{n}$. In this case, by Eq. (4.11), $g_j \in C_{-\varepsilon_j} g_j^{-1} b C_{\varepsilon_{j+1}}$.

By Eqs. (4.12) and (4.10), $\varepsilon_{j+1} = -\varepsilon_{-j-1-k} = -\varepsilon_j$. Since $b \in C_{-\varepsilon_j}$ then $b \in C_{\varepsilon_{j+1}}$. Consequently, $g_j \in C_{-\varepsilon_j} g_j^{-1} C_{\varepsilon_{j+1}}$. Since $C_{-\varepsilon_j} g_j^{-1} C_{\varepsilon_{j+1}} \subset (A \cup B) g_j (A \cup B)$, this is a contradiction. □

5 Oriented non-separating simple loops

This section is the "non-separating version" of Sect. 3. The main purpose here is the proof of Theorem 5.3, which describes the terms of the bracket of a simple non-separating conjugacy class and an arbitrary conjugacy class (see Fig. 5).

We will start by proving some elementary auxiliary results.

Lemma 5.1 *Let λ be a non-separating simple curve on Σ and p a point in λ. There exists a map $\psi : \mathbb{S}^1 \times [0,1] \longrightarrow \Sigma$, such that:*

(1) *There exists a point $q \in \mathbb{S}^1$, $\psi(q, 0) = \psi(q, 1) = p$.*
(2) *ψ is injective on $(\mathbb{S}^1 \times [0,1]) \setminus \{(q,0), (q,1)\}$*
(3) *$\psi|_{\mathbb{S}^1 \times \{0\}} = \lambda$.*

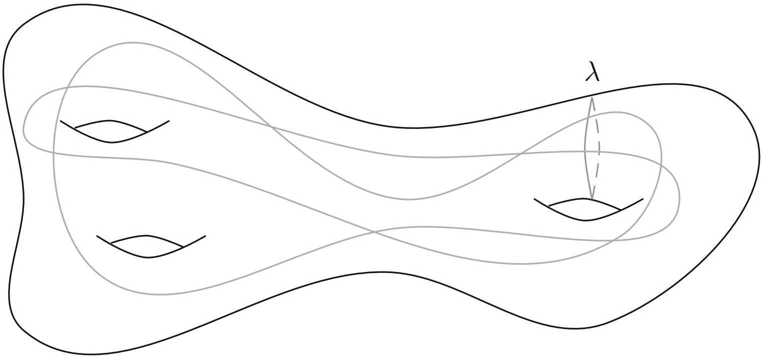

Fig. 5 The intersection of a non-separating curve λ and another curve

Proof (see Fig. 4) Choose a simple trivial curve τ such that $\tau \cap \lambda = \{p\}$ and the intersection of τ and λ is transversal. (The existence of such a curve is guaranteed by the following argument of Poincaré: Take a small arc β crossing λ transversally. Since $\Sigma \setminus (\beta \cup \lambda)$ is connected there exists an arc in $\Sigma \setminus (\beta \cup \lambda)$, with no self-intersections, joining the endpoints of β.)

Consider an injective map $\eta\colon \mathbb{S}^1 \times [0,1] \longrightarrow \Sigma$ such that $\lambda = \eta(\mathbb{S}^1 \times \{0\})$. Let $q \in \mathbb{S}^1$ be such that $\eta(q,0) = p$. Denote by C the image of the cylinder $\eta(\mathbb{S}^1 \times [0,1])$. Denote by ξ the boundary component of C defined by $\eta(\mathbb{S}^1 \times \{1\})$. Modify η if necessary so that τ intersects ξ in a unique double point s. (See Fig. 6).

Choose two points on s_1 and s_2 on ξ close to s and at both sides of s. Choose a embedded arc in the interior of C, intersecting τ exactly once, from s_1 to s_2 and denote it by κ. Denote by D the closed half disk bounded by κ and the subarc of ξ containing s.

Choose two embedded arcs α_1 and α_2 on Σ from p to s_1 and from p to s_2, intersecting only at p, and such that $\alpha_1 \cap \tau = \alpha_2 \cap \tau = \{p\}$, $\alpha_1 \cap C=\{p, s_1\}$ and $\alpha_2 \cap C=\{p, s_2\}$.

Consider the triangle T, with sides α_1, the arc in τ from s_1 to s_2 and α_2.

Denote by η_1 the restriction of η to $\eta^{-1}(D)$, $\eta_1\colon \eta^{-1}(D) \longrightarrow D$. Now take a homeomorphism $\eta_2\colon D \longrightarrow D \cup T$ such that $\eta_2|_\kappa$ is the identity and $\eta_2(s) = p$.

For each $x \in \mathbb{S}^1 \times [0,1]$, define $\psi\colon \mathbb{S}^1 \times [0,1] \longrightarrow \Sigma$ as $\eta(x)$ if $x \notin D$ and as $\eta_2\eta(x)$ otherwise. This map satisfies the required properties. □

Let ψ be the map of Lemma 5.1. Denote by λ_1 the curve $\psi(\mathbb{S}^1 \times \{1\})$. The homeomorphism $\vartheta\colon \lambda = \psi(\mathbb{S}^1 \times \{0\}) \longrightarrow \lambda_1 = \psi(\mathbb{S}^1 \times \{1\})$ defined by $\vartheta(\psi(s,0)) = \psi(s,1)$ induces an isomorphism $\varphi\colon \pi_1(\lambda, p) \longrightarrow \pi_1(\lambda_1, p)$. Denote by Σ_1 the subspace of Σ defined by $\Sigma \setminus \psi(\mathbb{S}^1 \times (0,1))$, and by τ the simple closed curve induced by the restriction of ψ to $\{q\} \times [0,1]$.

Lemma 5.2 *With the above notation, the fundamental group $\pi_1(\Sigma, p)$ of Σ is isomorphic to the HNN extension of $\pi_1(\Sigma_1, p)$ relative to φ. Moreover, τ is a representative of the element denoted by the stable letter t and if $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots g_{n-1}, t^{\varepsilon_n})$ is a cyclically reduced sequence of the HNN extension then there exists a sequence of curves $(\gamma_0, \gamma_1, \dots, \gamma_{n-1})$ such that for each $i \in \{0, 1, \dots, n-1\}$,*

(1) *The basepoint if γ_i is p.*
(2) *The curve γ_i is a representative of g_i.*
(3) *The inclusion $\gamma_i \subset \Sigma_1$ holds.*
(4) *The basepoint p is not a self-intersection point of γ_i. In other words, γ_i passes through p exactly once.*

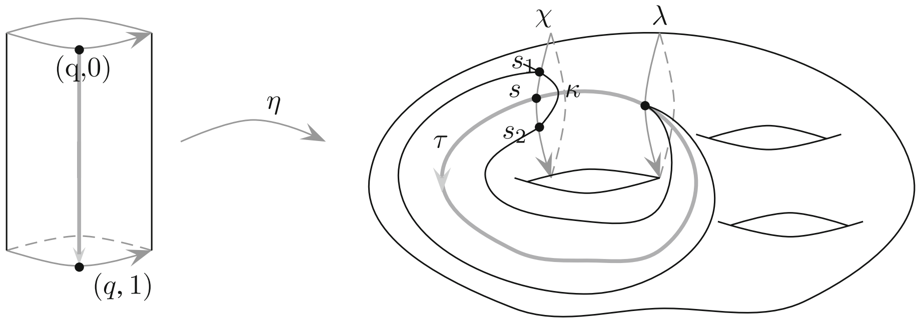

Fig. 6 The proof of Lemma 5.1

Proof By van Kampen's Theorem (see, for instance [16]), since $\Sigma_1 \cap \tau = \{p\}$ we have that $\pi_1(\Sigma_1 \cup \tau, p)$ is the free product of $\pi_1(\Sigma_1, p)$ and the infinite cyclic group $\pi_1(\tau, p)$.

Denote by D the disk $\psi((\mathbb{S}^1 \setminus q) \times (0, 1))$. Glue the boundary of D to the boundary of $\Sigma_1 \cup \tau$ as follows: attach λ, τ, λ_1 to $\mathbb{S}^1 \times \{0\}$, $q \times [0, 1]$, and $\mathbb{S}^1 \times \{1\}$ respectively. (The reader can easily deduce the orientations.)

The relation added by attaching the disk D shows that the $\pi_1(\Sigma, p)$ is isomorphic to the HNN extension of $\pi_1(\Sigma_1, p)$ relative to φ. Notice also that τ is a representative of t.

For each $i \in \{0, 1, \ldots, n-1\}$, let γ_i be a loop in Σ_1, based at p and representing g_i. By modifying these curves by a homotopy relative to p if necessary, we can assume that each of them intersects p exactly once, as desired. Then (4) holds and the proof is complete. □

Recall that there is a canonical isomorphism between free homotopy classes of curves on a surface Σ and conjugacy classes of elements of $\pi_1(\Sigma)$. From now on, we will identify these two sets.

The following theorem gives a combinatorial description of the bracket of two oriented curves, one of them simple and non-separating (see Fig. 5).

Theorem 5.3 *Let* λ *be a separating simple closed curve. Let* $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \ldots, g_{n-1}, t^{\varepsilon_n})$ *be a cyclically reduced sequence for the HNN extension of Lemma 5.2 determined by* λ*. Let* y *be the element of* $\pi_1(\Sigma, p)$ *associated with* λ*. Then the following holds.*

(1) *There exists a representative* η *of the conjugacy class of the product* $g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{n-1}$ *such that* η *and* λ *intersect transversally at* p *with multiplicity* n*.*
(2) *There exists* $s \in \{1, -1\}$ *such the bracket is given by*

$$s[g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{n-1}t^{\varepsilon_n}, y] = \sum_{i:\varepsilon_i=1} g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{i-1}yt^{\varepsilon_i}g_i\cdots g_{n-1}t^{\varepsilon_n} - \sum_{i:\varepsilon_i=-1} g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{i-1}\varphi(y)t^{\varepsilon_i}g_i\cdots g_{n-1}t^{\varepsilon_n}$$

Proof Let $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1})$ denote a sequence of curves obtained in Lemma 5.2 for the sequence $(g_0, t^{\varepsilon_1}, g_1,t^{\varepsilon_2} \ldots, g_{n-1},t^{\varepsilon_n})$.

Let $D \subset \Sigma$ be a small disk around p. Observe that $D \cap \psi(\mathbb{S}^1 \times [0, 1])$ consists in two "'triangles" T_1 and T_2, intersecting at p (see Fig. 7). Two of the sides of one of these triangles are subarcs of λ. Denote this triangle by T_1. Suppose that the beginning of τ is inside T_1 (the proof for the other possibility is analogous.)

For each $i \in \{1, 2, \ldots, n\}$, if $\varepsilon_i = 1$, τ_i will denote a copy of the curve τ and if $\varepsilon_i = -1$, τ_i will denote a copy of the curve τ with opposite direction. Denote by γ the curve $\gamma_1\tau_1\gamma_2\tau_2\cdots\gamma_n\tau_n$. Clearly γ is a representative of $g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{n-1}t^{\varepsilon_n}$.

The intersection of γ and λ consists in $2n$ points, located at the beginning and end of τ_i for each $i \in \{1, 2, \ldots, n\}$.

It is not hard to see that for each $i \in \{1, 2, \ldots, n\}$ if $\varepsilon_i = 1$, the intersection point of γ and λ located at the end of τ_i can be removed by a small homotopy. Similarly, if $\varepsilon_i = -1$ then the intersection point at the beginning of τ_i can be removed by a small homotopy. (See for example Fig. 7, where the two arcs at the of τ and beginning of γ_1 are replaced by the arc ρ_i.)

Denote by η the curve obtained after homotoping γ to remove the n removable points. Note that η intersects λ at p with multiplicity n. More specifically, for each $i \in \{1, 2, \ldots, n\}$, if $\varepsilon_i = 1$ then there is an intersection at the beginning of τ_i, and if $\varepsilon_i = -1$ there in an intersection at the end of τ_i. Since η crosses λ these intersections can be taken to be transversal. Thus (1) is proved.

Now we will compute the bracket $[g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{n-1}t^{\varepsilon_n}, y]$ using η and λ as representatives. Since η and λ have n intersection points, there are n terms.

Let $i \in \{1, 2, \ldots, n\}$. Assume first that $\varepsilon_i = 1$. The term of the bracket corresponding to this intersection point is obtained by inserting λ between γ_{i-1} and τ_i. Since the transformations we applied to γ to obtain η can be now reversed, then the free homotopy class of this term is

$$g_0t^{\varepsilon_1}g_1\cdots g_{i-1}yt^{\varepsilon_i}g_i\cdots g_{n-1}t^{\varepsilon_n}.$$

Assume now that $\varepsilon_i = -1$. The term of the bracket corresponding to this intersection point is obtained by inserting λ right after τ_i. This yields the element

$$g_0t^{\varepsilon_1}g_1\cdots g_{i-1}t^{\varepsilon_i}yg_i\cdots g_{n-1}t^{\varepsilon_n}.$$

By using the relation $t^{-1}y = \varphi(y)t^{-1}$ this element can be written as

$$g_0t^{\varepsilon_1}g_1\cdots g_{i-1}\varphi(y)t^{\varepsilon_i}g_i\cdots g_{n-1}t^{\varepsilon_n}.$$

To conclude, observe that pairs of terms corresponding to $\varepsilon_i = 1$ and $\varepsilon_i = -1$ have opposite signs because the tangents of η at the corresponding points point in opposite directions, and the tangent of λ is the same for both terms. (see Fig. 8) □

6 Some results on surface groups

This section contains auxiliary results showing that certain equations do not hold in the fundamental group of the surface. These results will be used in Sects. 7 and 8 to prove that certain sequences satisfy the hypothesis of Theorems 2.12, 2.14, 4.16 and 4.19.

We start with a well known result included for completeness.

Lemma 6.1 *If g is a non-trivial element of a free group G, then g and g^{-1} are not conjugate.*

Proof Suppose that g and g^{-1} are conjugate. Then there exists an element x of G, conjugate to g which can be represented by a cyclically reduced word $x_1x_2\ldots x_n$ in the free generators of G. Since x and x^{-1} are conjugate, there exist an integer k such that $x_1x_2\ldots x_n = \overline{x}_{k-1}\overline{x}_{k-2}\ldots\overline{x}_{k-n}$. Thus for each $i \in \{1, 2, \ldots, n\}$, $x_i = \overline{x}_{k-i}$.

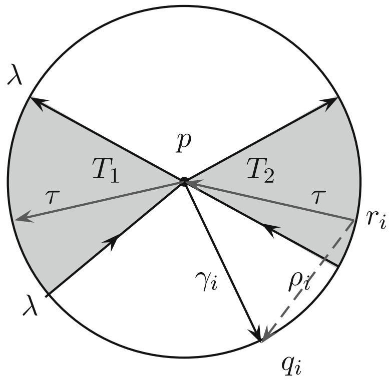

Fig. 7 Proof of Theorem 5.3

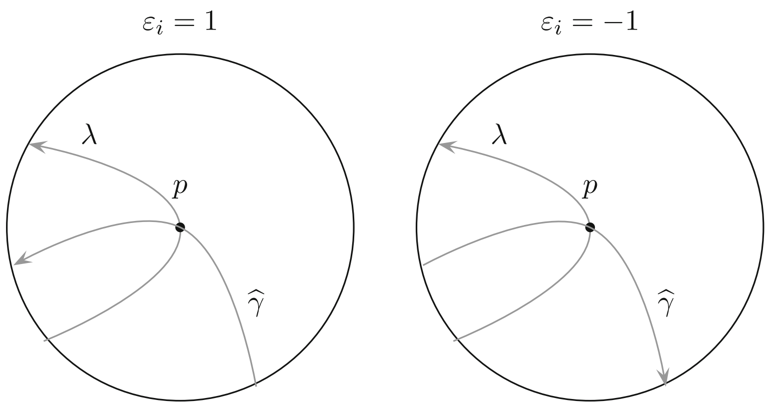

Fig. 8 Proof of Theorem 5.3

If k is even, $x_{\frac{k}{2}} = \overline{x}_{\frac{k}{2}}$, a contradiction. Therefore k is odd. Thus $x_{\frac{k-1}{2}} x_{\frac{k+1}{2}} = x_{\frac{k}{2}} \overline{x}_{\frac{k-1}{2}}$ which implies the word x is not reduced. Hence the proof is complete. □

Proposition 6.2 *Let Σ be an orientable surface with non-empty boundary. Let a be an element of $\pi_1(\Sigma)$ which can be represented by a simple closed curve parallel to a boundary component of Σ. Then the cyclic group generated by a is malnormal in $\pi_1(\Sigma)$.*

Proof Suppose that the cyclic group generated by a is not malnormal. Then there exist integers m and n and $g \in \pi_1(\Sigma) \setminus \{a^k, k \in \mathbb{Z}\}$ such that the equation $a^m g a^n g^{-1} = 1$ holds. Hence a^{-m} and a^n are conjugate. Write a as xvx^{-1} where x and v are words in the free generators of $\pi_1(\Sigma)$ and v is cyclically reduced. The words v^n and v^{-m} are cyclically reduced and conjugate. Since $\pi_1(\Sigma)$ is a free group, if two cyclically reduced words are conjugate then one is a cyclic permutation of the other. In particular, two cyclically reduced conjugate words have the same length. Thus $n = m$ or $m = -n$. By Lemma 6.1, $m = -n$.

Therefore a^m and g, are two elements of a free group which commute. Hence a^m and g are powers of the same element $c \in F$ (see [25, page 10] for a proof of this statement.) Let k be an integer such that $a = c^k$. By hypothesis, a is not a proper power; thus $k \in \{1, -1\}$. Consequently, either $c = a$ or $c = a^{-1}$. This implies that g is a power of a. □

Proposition 6.3 *L Let Σ_1 and φ be as in Lemma 5.2. If Σ_1 is not a cylinder then the HNN extension of Lemma 5.2 is separated. (See Definition 4.14).*

Proof Let λ and λ_1 be as in the paragraph before Lemma 5.2. Let a and b denote elements on the fundamental group of Σ_1 such that λ and λ_1 are representatives of a and b respectively.

Assume that the HNN extension is not separated. Then there exist non-zero integers m and n and $g \in \pi_1(\Sigma_1)$ such that the equation $a^m g b^n g^{-1} = 1$ holds. Denote by $\widetilde{\Sigma}_1$ the surface obtained by collapsing the boundary component of Σ_1 associated with a. Denote by $\widetilde{b}$ the element of $\pi_1(\widetilde{\Sigma}_1)$ induced by b. Observe that $\widetilde{b}^n = 1$. Since $n \neq 0$, by [25, Proposition 2.16], $\widetilde{b} = 1$. Then $\widetilde{\Sigma}_1$ is a disk. Therefore Σ_1 is a cylinder, a contradiction. □

Proposition 6.4 *Let Σ be an orientable surface with non-empty boundary which is not the cylinder. Let p be a point in Σ. Let a, b and g be elements of $\pi_1(\Sigma)$. Assume that a and b can be represented by simple closed curves freely homotopic to boundary components of Σ. Moreover assume that either a and b are either equal or not conjugate. If g is neither a power of a nor of b then for every pair of integers n and m, $ga^m gb^n \neq 1$.*

Proof Assume that $ga^m gb^n = 1$. Notice that $m \neq 0$ and $n \neq 0$. Let $u = ga^m$. Since $ga^m gb^n = ga^m ga^m a^{-m} b^n = u^2 a^{-m} b^n$,

$$u^2 = b^{-n} a^m. \tag{6.1}$$

Suppose that Σ has three or more boundary components or Σ has two boundary components and $a = b$. Then there exists a free basis of the fundamental group of Σ containing a and b. (In the two considered cases, $a = b$ is possible) Eq. (6.1) does not hold.

Suppose that Σ has two boundary components and $a \neq b$. Denote by h the genus of Σ. Since Σ is not the cylinder, $h > 1$. Then there exists a presentation of the fundamental group of Σ such that the free generators are $a, a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_h$ and

$$b = aa_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_hb_ha_h^{-1}b_h^{-1} \tag{6.2}$$

Combining Eqs. (6.1) and (6.2) we obtain

$$u^2 = (aa_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_hb_ha_h^{-1}b_h^{-1})^{-n}a^m.$$

Observe that all the elements of the right hand side of the above equation are in the free generating set of the group. We can check that both assumptions $n > 0$ and $n < 0$ lead to a contradiction. Since $n \neq 0$ the result is proved in this case.

Finally, suppose that Σ has one boundary component and $a = b$. Then there exists a presentation of the fundamental group of Σ with free generators $a_1, b_1, \ldots, a_h, b_h$ such that

$$a = a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_hb_ha_h^{-1}b_h^{-1} \tag{6.3}$$

If $m - n$ is even then $u = a^{\frac{m-n}{2}}$ which implies that g is a power of a. Hence $m - n$ is odd. In this case, it is easy to check that the equation $u^2 = a^{m-n}$ does not hold. □

7 Goldman Lie algebras of oriented curves

In this section we combine some of our previous results to prove Theorem 7.7.

Definition 7.1 Let x and y be conjugacy classes of $\pi_1(\Sigma)$ such that x can be represented by a simple loop. We associate a non-negative integer $t(x, y)$ to x and y, called the *number of terms of y with respect to x* in the following way:

Firstly, assume that x has a separating representative. Let $(w_1, w_2, \ldots, w_n)$ be cyclically reduced sequence for the amalgamated product of Remark 3.1 such that the product $w_1 w_2 \cdots w_n$ is conjugate to y. (The existence of such a sequence is guaranteed by Theorem 2.7.) We define $t(x, y) = 0$ if $n \leq 1$ and $t(x, y) = n$ otherwise. (By Theorem 2.7, $t(x, y)$ is well defined if x has a separating representative.)

Secondly, assume that x can be represented as a non-separating simple closed curve. Let $(g_0, t^{\varepsilon_1}, g_1 \ldots, g_{n-1}, t^{\varepsilon_n})$ be a cyclically reduced sequence for the HNN extension defined in Lemma 5.2 such that the product of this sequence is conjugate to y. (The existence of such a sequence is guaranteed by Theorem 4.5.) We set $t(x, y) = n$. (By Theorem 4.5, $t(x, y)$ is well defined in this case.)

Let α and β be two curves that intersect transversally. The *geometric intersection number of* α *and* β is the number of times that α crosses β. More precisely, the geometric intersection number of α and β is the number of pair of points (u, v), where u is in the domain of α, v is in the domain of β, u and v have the same image in Σ and the branch through u is transversal to the branch through v in the surface. Thus the geometric intersection number of α and β is the number of intersection points of α and β counted with multiplicity.

Let a and b denote two free homotopy classes of curves. The *minimal intersection number of* a *and* b, denoted by $i(a, b)$, is the minimal possible geometric intersection number of pairs of curves representing a and b.

Lemma 7.2 *Let x and y be conjugacy classes of $\pi_1(\Sigma)$. Assume that x can be represented as a simple closed curve. Then, $i(x, y) \leq t(x, y)$.*

Proof If x can be represented by a separating (resp. non-separating) curve, by Lemma 3.2 (resp. Theorem 5.3) there exist representatives of x and y with exactly $t(x, y)$ intersection points. □

Remark 7.3 If x and y are conjugacy classes of $\pi_1(\Sigma)$ and x can be represented as a simple closed curve x, it can be proved directly that $i(x, y) = t(x, y)$. Since this equality follows from Theorem 7.7, we do not give a proof of this statement here.

Definition 7.4 Let u and v be conjugacy classes of $\pi_1(\Sigma)$. The *number of terms of the Goldman Lie bracket* $[u, v]$ denoted by $g(u, v)$ is the sum of the absolute values of the coefficients of the expression of $[u, v]$ in the basis of the vector space given by the set of conjugacy classes.

Remark 7.5 Let u and v be conjugacy classes of $\pi_1(\Sigma)$. Since one can compute the Lie bracket by taking representatives of u and v with minimal intersection, and the bracket may have cancellation, $g(u, v) \leq i(u, v)$.

Our next result states that in the closed torus, the Goldman bracket always "counts" the intersection number of two free homotopy classes.

Lemma 7.6 *Let a and b denote free homotopy classes of the fundamental group of the closed oriented torus $\mathbb{T}$. Then $i(a, b) = g(a, b)$. Moreover, $[a, b] = \pm i(a, b)\ \ a \cdot b$, where $a \cdot b$ denotes the based loop product of a representative of a with a representative of b.*

Proof Assume first that neither a nor b is a proper power. Thus a admits a simple representative α, and there exists a class c which admits a simple representative δ such that δ intersects α exactly once with intersection number $+1$. Whence, a and c are a basis of fundamental group of $\mathbb{T}$.

Let k and l be integers such that $b = a^k c^l$. The universal cover of $\mathbb{T}$ is the euclidean plane $\mathbb{R}^2$. We can consider a projection map $p: \mathbb{R}^2 \longrightarrow \mathbb{T}$ such that the liftings of α are the horizontal lines of equation $y = n$ with $n \in \mathbb{Z}$ and the liftings of δ are vertical lines with equation $x = n$ with $n \in \mathbb{Z}$. Thus there exists a representative β of b such that the liftings of β are lines of equation $y = \frac{l}{k}x + n$ with $n \in \mathbb{Z}$.

One can check that the intersection of α and δ is exactly the projection of $\{(0, 0), (\frac{k}{l}, 0), (2\frac{k}{l}, 0), \dots, ((l-1)\frac{k}{l}, 0)\}$. Moreover, all these points project to distinct points and the sign of the intersection of α and δ at each of these points is equal to the sign of l.

Thus $[a, b] = [a, a^k c^l] = l \ \ a^{k+1}c^l$. Thus $g(a, b) = |l|$. Since there are representatives of a and b intersecting in l points, $i(a, b) \leq g(a, b)$. By Remark 7.5, $i(a, b) = g(a, b)$.

Assume now a and b are classes which are proper powers. Thus there exist simple closed curves ϵ and γ and integers i and j such that ϵ^i is a representative of a and γ^j is a representative of b. Moreover by the first part of this proof, we can assume that if e denotes the free homotopy class of ϵ and g denotes the free homotopy class of γ, then ϵ and γ intersect in exactly $i(e, g)$ points. Take i "parallel" copies of ϵ very close to each other and reconnect them so that they form a representative α of a. Do the same with j copies of γ, and denote by β the representative of b we obtain. We can perform this surgery far from the intersection points of ϵ and γ so that the number of intersection points of α and β is exactly $i \cdot j \cdot i(\epsilon, \gamma)$, whence $(i \cdot j \cdot i(e, g)) \leq i(a, b)$. On the other hand, the bracket is $[a, b] = \pm(i \cdot j \cdot i(e, g)) \ \ (a \cdot b)$. By Remark 7.5, $g(a, b) = i(a, b)$, as desired. □

Here is a sketch of the proof of our next result (the detailed proof follows the statement): Given a free homotopy class x with a simple representative and an arbitrary class y, we can write the bracket $[x, y]$ as in Theorem 3.4 or as in Theorem 5.3. The conjugacy classes of the terms of this bracket have representatives as the ones studied in Theorem 2.12 or in Theorem 4.16. By Proposition 6.2 (resp. Proposition 6.4) the free product on amalgamation (resp. the HNN extension) we are considering satisfies the hypothesis of Theorem 2.12 (resp. Theorem 4.16.) This implies that the terms of the bracket do not cancel.

Theorem 7.7 *Let x be a free homotopy class that can be represented by a simple closed curve on Σ and let y be any free homotopy class. Then the following non-negative integers are equal:*

(1) *The minimal number of intersection points of x and y.*
(2) *The number of terms in the Goldman Lie bracket $[x, y]$, counted with multiplicity.*
(3) *The number of terms in the reduced sequence of y determined by x.*

In symbols, $i(x, y) = g(x, y) = t(x, y)$.

Proof By Remarks 7.2 and 7.5, $g(x, y) \leq i(x, y) \leq t(x, y)$. Hence it is enough to prove that $t(x, y) \leq g(x, y)$. By Lemma 7.6, we can assume that Σ is not the torus.

We first prove that $t(x, y) \leq g(x, y)$ when x can be represented by a separating simple closed curve χ. By Theorem 2.7, there exists a cyclically reduced sequence $(w_1, w_2, \dots, w_n)$ for the free product of amalgamation determined by χ in Remark 3.1 such that the product $w_1 w_2 \cdots w_n$ is a representative of y. If $n = 0$ or $n = 1$, then $t(x, y) = 0$ and the result holds. If $n > 1$, by Theorem 3.4, there exists $s \in \{1, -1\}$ such that

$$s \ [x, y] = \sum_{i=1}^{n} (-1)^i w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n. \tag{7.1}$$

If i and j are such that there is cancellation between the i-th term and the j-th term of the right hand side of the Eq. (7.1), then $(-1)^i = -(-1)^j$. Consequently, i and j have different parity.

We will work at the basepoint indicated above and will abuse notation by pretending x and y are elements of the fundamental group of Σ (see Notation 3.3.) By Proposition 6.2, the cyclic group generated by x is malnormal in $\pi_1(\Sigma_1)$ and is malnormal in $\pi_1(\Sigma_2)$, where Σ_1 and Σ_2 are as in Remark 3.1. Thus the hypotheses of Theorem 2.12 hold for this free product with amalgamation. Hence by Theorem 2.12, there is no cancellation in the sum of the right hand side of Eq. (7.1). Consequently, $t(x, y) = g(x, y)$.

Now we prove the result when x can be represented by a non-separating simple closed curve λ. Consider the HNN extension of Lemma 5.2 determined by λ. By Theorem 4.5 there exists a cyclically reduced sequence $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$ such that the product $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n}$ is a representative of y. By Theorem 5.3, there exists $s \in \{1, -1\}$ such that $s[x, y]$ equals

$$\sum_{i:\varepsilon_i=1} g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{i-1} x t^{\varepsilon_i} g_i \cdots g_{n-1} t^{\varepsilon_n} - \sum_{i:\varepsilon_i=-1} g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{i-1} \varphi(x) t^{\varepsilon_i} \cdots g_{n-1} t^{\varepsilon_n}.$$

If $t(x, y) > g(x, y)$ then there is cancellation in the above sum. Therefore there exist two integers h and k such that $\varepsilon_h = 1$, $\varepsilon_k = -1$ and the products

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{h-1} x t^{\varepsilon_h} t^{\varepsilon_{h+1}} \cdots g_{n-1} t^{\varepsilon_n} \quad \text{and} \quad g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{k-1} \varphi(x) t^{\varepsilon_k} \cdots g_{n-1} t^{\varepsilon_n}$$

are conjugate. We use now the notations of Lemma 5.2 and the paragraph before Lemma 5.2. By Proposition 6.2, $\pi_1(\lambda, p)$ and $\pi_1(\lambda_1, p)$ are malnormal in $\pi_1(\Sigma_1, p)$. Since we are assuming that Σ is not a torus, then Σ_1 is not a cylinder. Then by Proposition 6.3 the HNN extension is separated. By Theorem 4.16 the two products above are not conjugate, a contradiction. Thus the proof is complete. □

Corollary 7.8 *If x and y are conjugacy classes of curves that can be represented by simple closed curves then $t(x, y) = t(y, x)$.*

8 Goldman Lie algebras of unoriented curves

Recall that Goldman [13] defined a Lie algebra of unoriented loops as follows. Denote by π^* the set of conjugacy classes of $\pi_1(\Sigma, p)$. For each $x \in \pi^*$, denote by $\overline{x}$ the conjugacy class of a representative of x with opposite orientation. Set $\widehat{x} = x + \overline{x}$ and $\widehat{\pi} = \{\overline{x} + x, x \in \pi^*\}$. The map $\widehat{\cdot}$ is extended linearly to the vector space of linear combinations of elements of π^*. Denote by V the real vector space generated by $\widehat{\pi}$, that is, the image of the map $\widehat{\cdot}$. For each pair of elements of π^*, x and y, define the unoriented bracket

$$[\widehat{x}, \widehat{y}] = \Big([x, y] + [\overline{x}, \overline{y}]\Big) + \Big([x, \overline{y}] + [\overline{x}, y]\Big) = \widehat{[x, y]} + \widehat{[x, \overline{y}]}.$$

Observe that we denote the bracket of oriented curves and the bracket of unoriented curves by the same symbol [,].

An *unoriented term* of the bracket $[a + \overline{a}, b + \overline{b}] = [\widehat{a}, \widehat{b}]$ of a pair of unoriented curves $\widehat{a} = a + \overline{a}$ and $\widehat{b} = b + \overline{b}$ is a term of the form $c(z + \overline{z})$, where c is an integer and z is a conjugacy class, that is, an element of the basis of V multiplied by an integer coefficient.

Let $u(x, y)$ denote the number of unoriented terms of the bracket $[\widehat{x}, \widehat{y}]$ considered as a bilinear map on V, counted with multiplicity. This is the sum of the absolute value of the coefficients of the expression of $[\widehat{x}, \widehat{y}]$ in the basis $\widehat{\pi}$.

Example 8.1 With the notations of Lemma 7.6,

$$[\widehat{a}, \widehat{b}] = [\widehat{a}, \widehat{a^k c^l}] = \pm i(a, b)(\widehat{a^{k+1} c^l} - \widehat{a^{k-1} c^l}).$$

Using the fact that every free homotopy class of curves in the torus admits a power of a simple closed curve as representative, we can show that for every pair of free homotopy classes x and y,

$$u(x, y) = 2 \cdot i(x, y) = 2 \cdot g(x, y) = 2 \cdot t(x, y).$$

The strategy of the proof of our next theorem is similar to that of Theorem 7.7, namely, we write the terms of the bracket in a certain form (using Theorem 3.4 or Theorem 5.3). By the results on Sect. 6, we can apply Theorems 2.14 and 4.19 to show that the pairs of conjugacy classes of these sums which have different signs are distinct by Theorems 2.14 and 4.19.

Theorem 8.2 *Let x and y be conjugacy classes of $\pi_1(\Sigma, p)$ such that x can be represented by a simple closed curve. Then the following non-negative integers are equal*

(1) *The number of unoriented terms of the bracket $[\widehat{x}, \widehat{y}]$, $u(x, y)$.*
(2) *Twice the minimal number of intersection points of x and y, $2 \cdot i(x, y)$.*
(3) *Twice the number of terms of the Goldman bracket on oriented curves, counted with multiplicity, $2 \cdot g(x, y)$.*
(4) *Twice the number of terms of the sequence of y with respect to x, $2 \cdot t(x, y)$.*

In symbols,

$$u(x, y) = 2 \cdot i(x, y) = 2 \cdot g(x, y) = 2 \cdot t(x, y).$$

Proof By Remark 7.5, $u(x, y) \leq 2 \cdot i(x, y) = 2 \cdot t(x, y)$. By Example 8.1, we can assume that Σ is not the torus. By Theorem 7.7, it is enough to prove that $u(x, y) = 2 \cdot t(x, y)$. Assume that $u(x, y) < 2 \cdot t(x, y)$.

By definition the bracket $[\widehat{x}, \widehat{y}]$ is an algebraic sum of terms of the form $\widehat{z} = z + \overline{z}$, where z is a conjugacy class of curves and z and $\overline{z}$ are terms of one of the four following brackets: $[x, y]$, $[x, \overline{y}]$, $[\overline{x}, y]$ and $[\overline{x}, \overline{y}]$. By Theorem 7.7, the number of terms of each of the four brackets above is $t(x, y)$. Since $u(x, y) < 2 \cdot t(x, y)$ there has to be one term belonging to one of above four brackets that cancels with a term of another of those brackets. Denote one of these terms that cancel by t_1 and the other one by t_2. If t_1 is a term of $[u, v]$, where $u \in \{x, \overline{x}\}$ and $v \in \{y, \overline{y}\}$ then t_2 is a term of of one of the following brackets: $[u, \overline{v}]$, $[\overline{u}, v]$, or $[\overline{u}, \overline{v}]$. Hence it suffices to assume that t_1 is a term of $[x, y]$ and to analyze each of the following three cases.

(1) t_2 is a term of $[\overline{x}, y]$
(2) t_2 is a term of $[x, \overline{y}]$
(3) t_2 is a term of $[\overline{x}, \overline{y}]$

Assume first that x can be represented by a separating curve χ. By Theorem 2.7, there exists a cyclically reduced sequence $(w_1, w_2, \dots w_n)$ in the amalgamating product of Remark 3.1 determined by χ such that the product $w_1 w_2 \cdots w_n$ is a representative of y. By Theorem 3.4, then there exists $s \in \{1, -1\}$ and $i, j \in \{1, 2, \dots, n\}$ such that $t_1 = s(-1)^i w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n$ and one of the following holds.

(1) $t_2 = s(-1)^{j+1} w_1 w_2 \cdots w_j x^{-1} w_{j+1} \cdots w_n$.
(2) $t_2 = s(-1)^{j+1} w_n^{-1} w_{n-1}^{-1} \cdots w_{j+1}^{-1} x w_j^{-1} \cdots w_1^{-1}$.
(3) $t_2 = s(-1)^j w_n^{-1} w_{n-1}^{-1} \cdots w_{j+1}^{-1} x^{-1} w_j^{-1} \cdots w_1^{-1}$.

(Note that when we change direction of one of the elements of the bracket, x or y, there is a factor (-1) because one of the tangent vectors at the intersection point has the opposite direction. Also, changing direction of both x and y does not change signs.)

Let us study first case (1). If t_1 and t_2 cancel then $(-1)^i = -(-1)^{j+1}$ and the products $w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n$ and $w_1 w_2 \cdots w_j x^{-1} w_{j+1} \cdots w_n$ are conjugate. Therefore i and j have equal parities. By Proposition 6.2, the subgroup generated by x is malnormal in the two amalgamated groups of the amalgamated product of Remark 3.1. (We are again treating x as an element of the fundamental group of the surface.) By Lemma 6.1, x and x^{-1} are not conjugate. Hence we can apply Theorem 2.12, with $a = x$ and $b = x^{-1}$ to show that $w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n$ and $w_1 w_2 \cdots w_j x^{-1} w_{j+1} \cdots w_n$ are not conjugate, a contradiction.

Similarly, by Theorem 2.14 and Proposition 6.4, Cases (2) and (3) are not possible.

Now assume that x has a non-separating representative. By Theorem 4.5 there exist a cyclically reduced sequence $(g_0, t^{\varepsilon_1}, g_1, \ldots g_{n-1}, t^{\varepsilon_n})$ whose product is an element of y. By Theorem 5.3 there exists an integer i such that the term t_1 has the form $s\varepsilon_i \cdot g_0 t^{\varepsilon_1} g_1 \cdots g_i u t^{\varepsilon_i} \cdots g_{n-1} t^{\varepsilon_n}$ where $u = x$ if $\varepsilon_i = 1$ and $u = \varphi(x)$ if $\varepsilon_i = -1$. By Theorem 5.3 there exist an integer j such that the term t_2 has one of the following forms.

(1) $t_2 = -s\varepsilon_j \cdot g_0 t^{\varepsilon_1} g_1 \cdots g_j v t^{\varepsilon_j} \cdots g_{n-1} t^{\varepsilon_n}$ where $v = x^{-1}$ if $\varepsilon_j = 1$ and $v = \varphi(x^{-1})$ if $\varepsilon_j = -1$.
(2) $t_2 = s\varepsilon_j \cdot g_{n-1}^{-1} t^{-\varepsilon_{n-1}} g_{n-2}^{-1} t^{-\varepsilon_{n-2}} \cdots g_j^{-1} v t^{-\varepsilon_j} \cdots g_1^{-1} t^{-\varepsilon_1} g_0^{-1} t^{-\varepsilon_n}$, where $v = x$ if $\varepsilon_j = -1$ and $v = \varphi(x)$ if $\varepsilon_j = 1$.
(3) $t_2 = -s\varepsilon_j \cdot g_{n-1}^{-1} t^{-\varepsilon_{n-1}} g_{n-2}^{-1} t^{-\varepsilon_{n-2}} \cdots g_j^{-1} v^{-1} t^{-\varepsilon_j} \cdots g_1^{-1} t^{-\varepsilon_1} g_0^{-1} t^{-\varepsilon_n}$ where $v = x$ if $\varepsilon_j = -1$ and $v = \varphi(x)$ if $\varepsilon_j = 1$.

The argument continues similarly to that of the separating case: By Proposition 6.2 and Proposition 6.3 the HNN extension is separated and the subgroups we are considering are malnormal. By Theorem 4.16 and Lemma 6.1, t_2 cannot have the form described in Case (1). Cases (2) and (3) are ruled out by Theorem 4.19 and Proposition 6.4. □

Let n be a positive integer and let x be a free homotopy class with representative χ. Denote by x^n the conjugacy class of the curve that wraps n times around χ. We can extend Theorem 8.2 to the case of multiple curves using the same type of arguments.

Theorem 8.3 *Let n be a positive integer and let x and y be conjugacy classes of $\pi_1(\Sigma, p)$ such that x can be represented by a simple closed curve χ. Then the following equalities hold.*

$$u(x^n, y) = n \cdot u(x, y) = 2i(x^n, y) = 2 \cdot n \cdot i(x, y) = 2 \cdot g(x^n, y) = 2 \cdot n \cdot g(x, y)$$
$$= 2 \cdot n \cdot t(x, y).$$

The next lemma is well known but we did not find an explicit proof in the literature. A stronger version of this result (namely, $i(x, x)$ equals twice the minimal number of self-intersection points of x) is proven in [9] for the case of surfaces with boundary.

Lemma 8.4 *If x is a homotopy class which does not admit simple representatives then the minimal intersection number of x and x is not zero. In symbols, $i(x, x) \neq 0$.*

Proof Let α and β be two transversal representatives of x. Let χ be a representative of x with minimal number of self-intersection points and let P be a self-intersection point of χ.

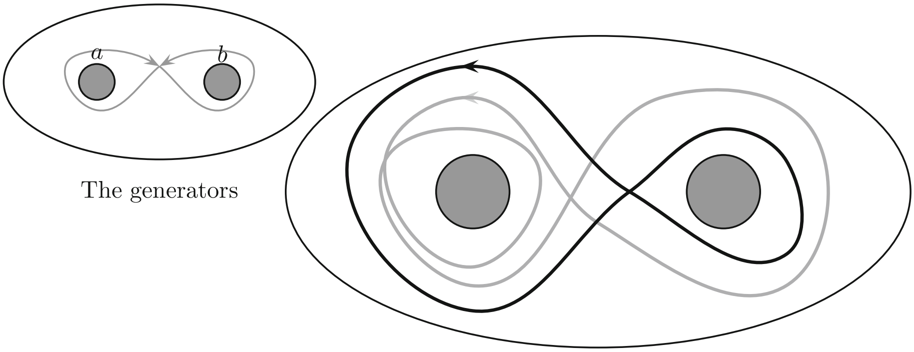

Fig. 9 An example of distinct essentially intersecting curves with zero bracket

Let $p\colon H \longrightarrow \Sigma$ be the universal cover of Σ. Consider two distinct lifts of χ, $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$ which intersect at a point Q of H such that $p(Q) = P$. Consider a lift of α, $\widetilde{\alpha}$ such that the endpoints of $\widetilde{\alpha}$ and χ_1 coincide. Analogously, consider a lift of β, $\widetilde{\beta}$ such that the endpoints of $\widetilde{\beta}$ and χ_2 coincide. Since χ intersects in a minimal number of points, the endpoints of the lifts χ_1 and χ_2 are linked. Thus the endpoints of α_1 and $\widetilde{\beta}$ are linked. Hence $\widetilde{\alpha}$ and $\widetilde{\beta}$ intersect in H. Consequently, α and β intersect in Σ. □

Our next result is a global characterization of free homotopy classes with simple representatives in terms of the Goldman Lie bracket.

Corollary 8.5 *Let x be a free homotopy class. Then x contains a power of a simple representative if and only if for every free homotopy class y the number of terms of the bracket of x and y is equal to the minimal intersection number of x and y. In symbols, x contains a power of a simple representative if and only if $g(x, y) = i(x, y)$ for every free homotopy class y.*

Proof If x contains a simple representative and y is an arbitrary free homotopy class then $g(x, y) = i(x, y)$ by Theorem 7.7. If x does not have a simple representative then by Lemma 8.4, $i(x, x) \geq 1$. On the other hand, by the antisymmetry of the Goldman Lie bracket we have $[x, x] = 0$. Thus $g(x, x) = 0$ and the proof of the corollary is complete. □

9 Examples

The assumption in Theorems 7.7 and 8.2 that one of the curves is simple cannot be dropped. Goldman [13] gave the following example, attributed to Peter Scott: For any conjugacy class a, the Lie bracket $[a, a] = 0$. On the other hand, if a cannot be represented by a power of a simple curve, then any two representatives of a cannot be disjoint.

Here is a family of examples:

Example 9.1 Consider the conjugacy classes of the curves *aab* and *ab* in the pair of pants (see Fig. 9.) The term of the bracket correspond to p_1 is the conjugacy class of *aabba* and the term of the bracket corresponding to p_2 is *baaab*.The conjugacy classes of both terms are the same, and the signs are opposite. Then the Goldman bracket of these conjugacy classes is zero. Nevertheless, the minimal intersection number is two.

More generally, for every pair of positive integers n and m, the curves $a^n b$ and $a^m b$ have minimal intersection $2\min(m, n)$. Nevertheless, the bracket of these pairs vanishes, that is, $[a^n b, a^m b] = 0$. (The intersection number, as well as the Goldman Lie bracket, can be computed using results in [3].)

10 Application: factorization of Thurston's map

Denote by $C(\Sigma)$ the set of all free homotopy classes of undirected curves on a surface Σ which admit a simple representative. Consider the map $\phi\colon C(\Sigma) \longrightarrow V^{C(\Sigma)}$, defined by $\phi(a)(b) = [a, b]$, using the notation of Sect. 8. For each (reduced) linear combination c of elements of the vector space W, define a map abs: $V \longrightarrow \mathbb{Z}_{\geq 0}$, where abs$(c)$ is the sum of the absolute values of the coefficients of c. The function abs is essentially an L^1 norm on V.

By Theorem 7.7 the composition abs $\circ\, \phi\colon C(\Sigma) \longrightarrow \mathbb{Z}_{\geq 0}^{C(\Sigma)}$ is (up to a non-zero multiple) the map defined by Thurston in [29] which he used to define the first mapping class group invariant compactification of Teichmüller space.

11 Application: decompositions of the vector space generated by conjugacy classes

For each w in W, the vector space of free homotopy classes of curves, the *adjoint map determined by* w, denoted by ad_w is defined for each $y \in W$ by $\mathrm{ad}_w(y) = [y, w]$.

Let a denote the conjugacy class of a closed curve on a surface Σ and let n be a non-negative integer. Denote by $W_n(a)$ the subspace of W generated by the set of conjugacy classes of oriented curves with minimal number of intersection points with a equal to n.

In this section we prove that if a is the conjugacy class of a simple closed curve then $W_n(a)$ is invariant under ad_a. Moreover, we give a further decomposition of $W_n(a)$ into subspaces invariant under ad_a.

11.1 The separating case

Let χ be a separating simple closed curve. Let $G *_C H$ be the amalgamated free product defined by χ in Remark 3.1. Let $(w_1, w_2, \ldots, w_n)$ be a cyclically reduced sequence for this amalgamated free product. Denote by $W(w_1, w_2, \ldots, w_n)$ the subspace generated by the conjugacy classes which have representatives of the form $v_1 v_2 \cdots v_n$ where for each i in $\{1, 2, \ldots, n\}$, v_i is an element of the double coset $C w_i C$.

Proposition 11.1 *With the notations above, we have*

(1) *The subspace $W_n(x)$ is the disjoint union of the subspaces $W(w_1, w_2, \ldots, w_n)$, where $(w_1, w_2, \ldots, w_n)$ runs over all cyclically reduced sequences of n terms of the free product with amalgamation determined by x.*
(2) *The subspaces $W(w_1, w_2, \cdots, w_n)$ and $W_n(x)$ are invariant under* ad_x*, the adjoint map determined by x.*

Proof First prove (1). Consider a conjugacy class $y \in W_n(x)$. By Theorem 7.7 there exists a cyclically reduced sequence $(w_1, w_2, \ldots, w_n)$ with n terms for the amalgamating product of Remark 3.1 determined by a representative of x with product in y. Hence $y \in W(w_1, w_2, \ldots, w_n)$.

Observe that for each $i \in \{1, 2, \ldots, n\}$, $(w_1, w_2, \ldots, w_i x, w_{i+1}, \ldots w_n)$ is a cyclically reduced sequence. Thus (2) is an immediate consequence of (1), Theorem 3.4 and the definition of the subspaces $W(w_1, w_2, \ldots, w_n)$. □

Remark 11.2 It is not hard to see that the subspaces $W(w_1, w_2, \ldots, w_n)$ are also invariant under the map induced by the Dehn twist around x. Indeed, using Lemma 3.2, it is not hard to see that the Dehn twist around x of the conjugacy class of $w_1 w_2 \cdots w_n$ can be represented by

$$w_1 x w_2 \overline{x} w_3 x w_4 \overline{x} \cdots w_n \overline{x} \quad \text{or} \quad w_1 \overline{x} w_2 x w_3 \overline{x} \cdots w_n x$$

where the choice between the two conjugacy classes above is determined by the orientation of the surface, the orientation of x and the subgroup to which w_1 belongs.

Question 11.3 Denote by X the cyclic group of automorphisms of W generated by ad_x. Let $(w_1, w_2, \ldots, w_n)$ be a cyclically reduced sequence. It is not hard to see that the subspace $W(w_1, w_2, \ldots, w_n)$ is invariant under X. Is $W(w_1, w_2, \ldots, w_n)$ the minimal (with respect to inclusion) subspace of W containing the conjugacy class of $w_1 w_2, \cdots, w_n$ and invariant under X?

11.2 The non-separating case

We now develop the analog of Subsect. 11.1 for the non-separating case. In the separating case, the terms of cyclic sequences of double cosets belong to alternating subgroups. In the non-separating case, there is a further piece of information, namely the sequence of ε's. Thus the arguments, although essentially the same, present some extra technical complications. Also, there are more invariant subspaces surfacing.

Let γ denote a separating curve. Let $G^{*\varphi}$ be the HNN extension constructed in Lemma 5.2 with γ. Let $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$ denote a cyclically reduced sequence. Denote by $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \ldots, g_{n-1}, \varepsilon_n)$ the subspace of W generated by the conjugacy classes which have representatives of the form $h_0 t^{\varepsilon_1} h_1 t^{\varepsilon_2} \cdots h_{n-1} t^{\varepsilon_n}$ where for each $i \in \{1, 2, \ldots, n\}$, h_i is an element of the double coset $C_{-\varepsilon_i} g_i C_{\varepsilon_{i+1}}$. Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be a sequence of integers such that for each $i \in \{1, 2, \ldots, n\}$, $\varepsilon_i \in \{-1, 1\}$. Denote by $W(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ the subspace of W generated by the conjugacy classes which have representatives of the form $h_0 t^{\varepsilon_1} h_1 t^{\varepsilon_2} \cdots h_{n-1} t^{\varepsilon_m}$ where $(h_0, \varepsilon_1, h_1, \varepsilon_2, \cdots, h_{n-1}, \varepsilon_n)$ is a cyclically reduced sequence.

The following result can be proved with arguments similar to those of the proof of Proposition 11.1, using Theorem 5.3 and Theorem 8.2.

Proposition 11.4 *With the above notations,*

(1) *The subspaces* $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \ldots, g_{n-1}, \varepsilon_n)$, $W(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ *and* $W_n(y)$ *are invariant under* ad_y, *the adjoint map determined by* y.
(2) *The subspace* $W_n(y)$ *is the disjoint union of subspaces of the form* $W(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ *where* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ *is a sequence of integers such that for each* $i \in \{1, 2, \ldots, n\}$, $\varepsilon_i \in \{-1, 1\}$.
(3) *The subspace* $W(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ *is a disjoint union of* $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \ldots, g_{n-1}, \varepsilon_n)$ *where* $(g_0, \varepsilon_1, g_1, \varepsilon_2, \ldots, g_{n-1}, \varepsilon_n)$ *runs over cyclically reduced sequences with a sequences of* ε*'s given by* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

Remark 11.5 It is not hard to see that the subspaces $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \ldots, g_{n-1}, \varepsilon_n)$ are also invariant under the map induced by the Dehn twist around y. Indeed, using Lemma 3.2, one sees that the Dehn twist around y of the conjugacy class of $g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$ (where $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})$ is a cyclically reduced sequence) is represented by one of the following products:

$$g_0 u_1 t^{\varepsilon_1} g_1 u_1 t^{\varepsilon_2} \ldots, g_{n-1} u_n t^{\varepsilon_n},$$

where one of the following holds.

1. For each $i \in \{1, 2, \cdots, n\}$, if $\varepsilon_i = 1$ then $u_i = y$ and if $\varepsilon_i = -1$ then $u_i = \varphi(\overline{y})$.
2. For each $i \in \{1, 2, \cdots, n\}$, if $\varepsilon_i = 1$ then $u_i = \overline{y}$ and if $\varepsilon_i = -1$ then $u_i = \varphi(y)$.

where the choice between (1) and (2) is determined by the orientation of the surface and the orientation of y.

12 Application: the mapping class group and the curve complex

Let Σ be a compact oriented surface. By $\Sigma_{g,b}$ we denote an oriented surface with genus g and b boundary components. If Σ is a surface, we denote by $\mathcal{MCG}(\Sigma)$ the *mapping class group of* Σ, that is, the set of isotopy classes of orientation preserving homeomorphisms of Σ. We study automorphisms of the Goldman Lie algebra that are related to the mapping class group. The first two results, Theorems 12.4 and 12.5, apply to all surfaces. The stronger result, Theorem 12.6, applies only to surfaces with boundary.

Now we recall the curve complex, defined by Harvey in [15]. The *curve complex* $\mathrm{C}(\Sigma)$ *of* Σ is the simplicial complex whose vertices are isotopy classes of unoriented simple closed curves on Σ which are neither null-homotopic nor homotopic to a boundary component. If $\Sigma \neq \Sigma_{0,4}$ and $\Sigma \neq \Sigma_{1,1}$ then a set of $k+1$ vertices of the curve complex is the 0- skeleton of a k-simplex if the corresponding minimal intersection number of all pairs of vertices is zero, that is, if every pair of vertices have disjoint representatives.

For $\Sigma_{0,4}$ and $\Sigma_{1,1}$ two vertices are connected by an edge when the curves they represent have minimal intersection (2 in the case of $\Sigma_{0,4}$, and 1 in the case of $\Sigma_{1,1}$). If $b \leq 3$ the complex associated with $\Sigma_{0,b}$ is empty.

The following isomorphism is a theorem of Ivanov [17] for the case of genus at least two. Korkmaz [20] proved the result for genus at most one and Luo [23] gave another proof that covers all possible genera. The mapping class group of a surface Σ is denoted by $\mathcal{MCG}(\Sigma)$. Our discussion below is based on the formulation of Minsky [27].

Theorem 12.1 (Ivanov-Korkmaz-Luo)

(1) *The natural map* $h\colon \mathcal{MCG}(\Sigma) \longrightarrow \mathrm{AutC}(\Sigma)$ *is an isomorphism in all cases except for* $\Sigma_{1,2}$ *where it is injective with index* 2 *image.*
(2) *(Luo, [23]) Any automorphism of* $C(\Sigma_{1,2})$ *preserving the set of vertices represented by separating loops is induced by a self-homeomorphism of the surface* $\Sigma_{1,2}$.

We need the following result from [5].

Theorem 12.2 (Chas - Krongold) *Let* Σ *be an oriented surface with non-empty boundary and let* x *be a free homotopy class of curves in* Σ. *Then* x *contains a simple representative if and only if* $\langle x, x^3\rangle = 0$, *where* x^3 *is the conjugacy class that wraps around* x *three times.*

Lemma 12.3 *Let* x *be a free homotopy class of oriented simple closed curves. Then* x *has a non-separating representative if and only if there exists a simple class* y *such that the minimal intersection number of* x *and* y *is equal to one.*

Proof If x has a non-separating representative, the existence of y with minimal intersection number equal to one can be proved as in the proof of Lemma 5.1. Conversely, if x contains a separating representative, then the minimal intersection number of x and any other class is even. □

Theorem 12.4 *Let Ω be a bijection on the set π^* of free homotopy classes of closed curves on an oriented surface. Suppose the following:*

(1) *Ω preserves simple curves.*
(2) *If Ω is extended linearly to the free $\mathbb{Z}$ module generated by π^* then Ω preserves the Goldman Lie bracket. In symbols $[\Omega(x), \Omega(y)] = \Omega([x, y])$ for all $x, y \in \pi^*$.*
(3) *For all $x \in \pi^*$, $\Omega(\overline{x}) = \overline{\Omega(x)}$.*

Then the restriction of Ω to the subset of simple closed curves is induced by an element of the mapping class group. Moreover, if $\Sigma \neq \Sigma_{1,2}$ then the restriction of Ω to the subset of simple closed curves is induced by a unique element of the mapping class group.

Proof Since $\Omega(\overline{x}) = \overline{\Omega(x)}$, Ω induces a bijective map $\widehat{\Omega}$ on $\widehat{\pi} = \{x + \overline{x} : x \in \pi^*\}$.

Since Ω preserves the oriented Goldman bracket and the "change of direction", then it preserves the unoriented Goldman bracket. Let x be a class of oriented curves that contains a simple representative and let y be any class.

Since $\widehat{\Omega}$ preserves the unoriented Goldman Lie bracket, by Theorem 8.2, the minimal intersection number of x and y equals the minimal intersection number of $\widehat{\Omega}(x)$ and $\widehat{\Omega}(y)$. Then $\widehat{\Omega} \in \mathrm{AutC}(\Sigma)$. Thus by Theorem 12.1(1), $\widehat{\Omega}$ is induced by an element of the mapping class. Moreover if the surface is not $\Sigma_{1,2}$ then Ω is induced by unique element of the mapping class group.

Now we study $\Sigma_{1,2}$. By Lemma 12.3, x is a separating simple closed curve, if and only if $\Omega(x)$ is separating. Thus $\widehat{\Omega}$ maps bijectively the set of unoriented separating simple closed curves onto itself. Hence by Theorem 12.1(2), $\widehat{\Omega}$ is induced by an element of the mapping class group. □

By arguments similar to those of Proposition 12.4, one can prove the following.

Theorem 12.5 *Let Γ be a bijection on the set $\widehat{\pi}$ of unoriented free homotopy classes of closed curves on an oriented surface. Suppose the following*

(1) *Γ preserves simple curves.*
(2) *If Γ is extended linearly to the free $\mathbb{Z}$ module generated by π^* then Γ preserves the unoriented Goldman Lie bracket. In symbols $[\Gamma(x), \Gamma(y)] = \Gamma([x, y])$ for all $x, y \in \widehat{\pi}$.*

Then the restriction of Γ to the subset of simple closed curves is induced by an element of the mapping class group. Moreover, if $\Sigma \neq \Sigma_{1,2}$ then the restriction of Γ to the set of free homotopy classes with simple representatives is induced by a unique element of the mapping class group.

Theorem 12.6 *Let Ω be a bijection on the set π^* of free homotopy classes of curves on an oriented surface with non-empty boundary. Suppose the following*

(1) *If Ω is extended linearly to the free $\mathbb{Z}$ module generated by π^* then $[\Omega(x), \Omega(y)] = \Omega([x, y])$ for all $x, y \in \pi^*$.*
(2) *For all x in π^*, $\Omega(\overline{x}) = \overline{\Omega(x)}$.*
(3) *For all x in π^*, $\Omega(x^3) = \Omega(x)^3$.*

Then the restriction of Ω to the set of free homotopy classes with simple representatives is induced by an element of the mapping class group. Moreover, if $\Sigma \notin \{\Sigma_{1,1}, \Sigma_{2,0}, \Sigma_{0,4}\}$ then Ω is induced by a unique element of the mapping class group.

Proof Let x be an oriented closed curve. By hypothesis, $[\Omega(x), \Omega(x)^3] = \Omega([x, x^3])$. Thus $[x, x^3] = 0$ if and only if $[\Omega(x), \Omega(x)^3] = 0$. Thus by Theorem 12.2, $\Omega(x)$ is simple if and only if x is simple. Then the result follows from Theorem 12.4. □

All these results "support" Ivanov's statement in [18]:

Metaconjecture *"Every object naturally associated with a surface S and having a sufficiently rich structure has* Mod(S) *as its group of automorphisms. Moreover, this can be proved by a reduction theorem about the automorphisms of* C(S)*."*

In this sense, the Goldman Lie bracket combined with the power maps, have a "sufficiently rich" structure.

13 Questions and open problems

Problem 13.1 Etingof [11] proved using algebraic tools that the center of Goldman Lie algebra of a closed oriented surface consists of the one dimensional subspace generated by the trivial loop. On the other hand, if the surface has non-empty boundary, it is not hard to see that linear combinations of conjugacy classes of curves parallel to the boundary components are in the center. Hence it seems reasonable to conjecture that the center consists of linear combinations of conjugacy classes of boundary components. It will be interesting to use the results of this work to give a complete characterization of the center of the Goldman Lie algebra.

If u is an element of the vector space associated with unoriented curves on a surface, and u is a linear combination of classes that each admit simple representative and is not in the center then u is not in the center of the Goldman Lie algebra. To prove this, we need to combine our results with the fact that given two different conjugacy classes a and b that admit simple representatives, there exists a simple conjugacy class c such that the intersection numbers $i(a, c)$ and $i(b, c)$ are distinct (see [12] for a proof that such c exists). Nevertheless, by results of Leininger [21] we know that this argument cannot be extended to the cases of elements of the base that only have self-intersecting representatives.

Problem 13.2 As mentioned in the Introduction, Abbaspour [1] studied whether a three manifold is hyperbolic by means of the generalized Goldman Lie algebra operations. He used free products with amalgamations for this study. We wonder if it would be possible to combine his methods with ours, to give a combinatorial description of the generalized String Topology operations on three manifolds. In this direction, one could study the relation between number of connected components of the output of the Lie algebra operations and intersection numbers.

Problem 13.3 We showed that subspaces $W(w_1, w_2, \ldots, w_n)$ defined in Subsect. 11.1 are ad_x-invariant. Let z be a representative of a conjugacy class in $W(w_1, w_2, \ldots, w_n)$. It would be interesting to define precisely and study the "number of twists around x" of the sequence $(\mathrm{ad}_x^n(z))_{n\in\mathbb{Z}}$ and how this number of twists changes under the action of ad_x. Observe also that the Dehn twist around x increases or decreases the number of twists at a faster rate. These problems are related to the discrete analog of Kerckhoff's convexity [19] found by Luo [24].

Problem 13.4 The Goldman Lie bracket of two conjugacy classes, one of them simple, has no cancellation. On the other hand, there are examples (for instance Example 9.1) of pairs of classes with bracket zero and non-zero minimal intersection number. How does one characterize topololgically pairs of intersection points for which the corresponding terms cancel? In other words, what "causes" cancellation? The tools to answer to this question may involve the study of Thurston's compactication of Teichmüller space in the context of Bonahon's work on geodesic currents [2].

Problem 13.5 Dylan Thurston [28] proved a suggestive result: Let m be a union of conjugacy classes of curves on an orientable surface and let s be the conjugacy class of a simple closed curve. Consider representatives M of m, S of s, which intersect (and self-intersect) in the minimum number of points. Denote by P one of the self-intersection points of M. Denote by M_1 and M_2 the two possible ways of smoothing the intersection at P. Denote by m_1 and m_2 respectively the conjugacy classes of M_1 and M_2. Then Dylan Thurston's result is

$$i(m, s) = \max(i(m_1, s), i(m_2, s)).$$

These two "smoothings" are the two local operations one makes at each intersection point to find a term of the unoriented Goldman Lie bracket (when the intersection point P is not a self-intersection point of a curve). It might be interesting to explore the connections of Dylan Thurston's result and our work.

Remark 13.6 In a subsequent work we will give a combinatorial description of the set of cyclic sequences of double cosets under the action of ad. Also, we will study under which assumptions the cyclic sequences of double cosets are a complete invariant of a conjugacy class.

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