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ON THE STRUCTURE OF THE ω-LIMIT SETS FOR CONTINUOUS MAPS OF THE INTERVAL

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We introduce the notion of the center of a point for discrete dynamical systems and we study its properties for continuous interval maps. It is known that the Birkhoff center of any such map has depth at most 2. Contrary to this, we show that if a map has positive topological entropy then, for any countable ordinal α, there is a point $x_\alpha \in I$ such that its center has depth at least α. This improves a result by [Sharkovskii, 1966].

1. Introduction

The (*Birkhoff*) *center* of a discrete dynamical system is defined to be the closure of the set of its recurrent points and so it contains all the information about its "recurrent objects". In this paper we introduce the notion of the *center of a point* for discrete dynamical systems, and we study its properties for continuous interval maps with positive topological entropy. In [Simó, 1997] the same notion for a special kind of vector fields was studied and results in the spirit of ours were obtained. Earlier, a question posed to A. R. D. Mathias on the depth of the center of a point led to papers [Mathias, 1995, 1996, 1997] on the subject.

Throughout this paper, $\mathbb{N}$ and $\mathbb{Z}^+$ will denote the set of positive and non-negative integers, respectively. Let X be a compact metric space and let $f : X \longrightarrow X$ be a continuous map. The *ω-limit set of* x, denoted by $\omega_f(x)$, is the set of points $y \in X$ for which there exists a sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ tending to infinity such that $\lim_{k\to\infty} f^{n_k}(x) = y$. For any ordinal α we define by transfinite induction

$$W_f^0(x) = w_f(x)\,,$$

$$W_f^{\alpha+1}(x) = \bigcup_{z\in w_f^\alpha(x)} \omega_f(z)\,,$$

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and, when α is a limit ordinal,

$$W_f^\alpha(x) = \bigcap_{\beta<\alpha} W_f^\beta(x)\,,$$

Since the ω-limit set of a point is closed and invariant, by transfinite induction it can be proved that $W_f^\beta(x) \supset W_f^\alpha(x)$, for each $\beta < \alpha$. Hence, there is an ordinal γ such that $W_f^\beta(x) = W_f^\gamma(x)$ for any $\beta > \gamma$. The set $W_f^\gamma(x)$ is defined to be the *center* of x and is denoted by $C_f(x)$. The first γ satisfying the above property is called the *depth of* x with respect to f and is denoted by $d_f(x)$. In [Mathias, 1995] it is shown that if X is a complete separable metric space then, for each $x \in X$, $d_f(x)$ is at most the first uncountable ordinal. However, we conjecture that the number $d_f(x)$ is always countable.

Remark. The above definitions are related to the notion of the center of a dynamical system. Indeed, if X is a compact metric space and f is a continuous map from X to itself then the <u>center of f</u> can also be defined as follows. Set $X_0 = \overline{\cup_{x\in X}\omega_f(x)}$, $X_{\alpha+1} = \overline{\cup_{z\in X_\alpha}\omega_f(z)}$ when $\alpha \geq 0$ is a nonlimit ordinal and $X_\alpha = \cap_{\beta<\alpha}X_\beta$ when α is a limit ordinal. Then, the center of f coincides with X_{α^*} where α^* is the first countable ordinal such that $X_{\alpha^*} = X_\beta$ for each $\beta > \alpha^*$ (see [Birkhoff & Smith, 1928]).

In [Mathias, 1997] it is proved that if X is a complete separable metric space then $C_f(x)$ is the union of the ω-limit sets of all recurrent points in $\omega_f(x)$ (recall that $z \in X$ is *recurrent* if and only if $z \in \omega_f(z)$). In [Mathias, 1996] it is shown that any countable ordinal can be the depth of the center of a point. This latter result is proved for the space of the sequences of five symbols with the full one-sided shift.

The goal of this paper is to study the center and the depth of the center when the space X is a closed interval of the real line and the map f has positive topological entropy. In the sequel, Ω denotes the first uncountable ordinal. Our main result is the following.

Theorem A. *Let I be a compact real interval. Then $f \in \mathcal{C}(I, I)$ has positive topological entropy if and only if there exists a point $x \in I$ and a family $\{\omega_t\}_{t\in I}$ of perfect sets such that*

(a) *For any t there exists an $x_t \in I$ such that $\omega_t = \omega_f(x_t)$,*
(b) *$\omega_t \subset \omega_s \subset \omega_f(x)$ and $\omega_t \neq \omega_s$, for any $t < s$,*
(c) *for any $\alpha < \Omega$ and any $t < s$ there exists $y \in \omega_s$ such that $C_f(y) = \omega_t$ and $d_f(y) \geq \alpha$.*

We conjecture that Condition (c) in the above theorem can be improved so that the depth of y is exactly α (see Theorem 3.11).

Remark. If $f \in \mathcal{C}(I, I)$ has zero topological entropy then it follows easily from [Sharkovskii, 1968] that $d_f(x) \in \{0, 1\}$ for each $x \in I$ (see also Theorem 6.5 of [Bruckner & Smítal, 1993] and its proof).

In the sixties, Sharkovskii stated the following result (see [Sharkovskii *et al.*, 1993, Theorem 3.11] for the most recent reference): *Assume that $f \in \mathcal{C}(I, I)$. Then the following conditions are equivalent*:

(a) *f has a periodic orbit of period different from a power of two.*
(b) *There is a countable number of invariant closed sets ordered linearly by inclusion and such that there are dense trajectories in each of these sets* (*this means that they are homeomorphic to the Cantor set*).
(c) *There exist an uncountable set $B \subset I$ such that the family $\{\omega_f(x)\}_{x\in B}$ is ordered linearly by inclusion.*

The equivalence between Statements (a) and (b) follows from [Sharkovskii, 1966]. On the other hand, the equivalence between Statements (a) and (c) is said to follow from [Sharkovskii, 1968] (see, for instance, [Sharkovskii *et al.*, 1993, Theorem 3.11] or [Sharkovskii, 1982]). Since Statement (a) holds if and only if the topological entropy $h(f)$ of f is positive (see [Misiurewicz, 1979]), Theorem A gives a stronger result than the equivalence between (a) and (b) of Sharkovskii's Statement. Nevertheless, (a) and (c) are not equivalent, as it is shown by the following theorem.

Theorem B. *There is a map $f \in \mathcal{C}(I, I)$ with zero topological entropy but having a family of points $\{x_t\}_{t\in[0,1]} \subset I$ such that $\omega_f(x_t)$ is a proper subset of $\omega_f(x_s)$ whenever $t < s$.*

Proof. By [Bruckner & Smítal, 1993] there is a map $f \in \mathcal{C}(I, I)$ with $h(f) = 0$ and such that, for some x, $\omega_f(x) = Q \cup \{T_n\}_{n=1}^\infty$, where Q is a Cantor set and T_n are pairwise disjoint sets of isolated points of $\omega_f(x)$. Moreover, any T_n can be enumerated as $T_n = \{t_{n,k}\}_{k=-\infty}^\infty$ with $f(t_{n,k}) = t_{n,k+1}$ and, for any $M \subset \mathbb{N}$, the set $Q \cup \cup_{n \in M} T_n$ is the ω-limit set of a point.

Now let D be a countable dense subset of $[0, 1]$ (e.g. the set of rationals from $[0, 1]$) and let $\{n_r\}_{r \in D}$ be an enumeration of the set $\mathbb{N}$ by numbers from D. Then, defining $Q_t = Q \cup (\cup_{r \leq t} T_{n_r})$ for any $t \in [0, 1]$, the proof is complete. ■

Remark. Sharkovskii pointed out that Theorem A (and its abstract version, Theorem 3.11 below) imply that the depth of the center of a dynamical system generated by a triangular map of the square can be any countable ordinal. This generalizes Statement (iii) of the theorem in [Kolyada & Sharkovskii, 1991]. The proof of this fact is easy. Let $I = [0, 1]$, and let $f \in \mathcal{C}(I, I)$ be a map which contains a homeomorphic copy of the shift on the space of sequences of two symbols. For instance take

$$f(x) = \begin{cases} 3x & \text{for } x \in [0, 1/3]\,, \\ 1 & \text{for } x \in [1/3, 2/3]\,, \text{ and} \\ -3x + 3 & \text{otherwise}\,. \end{cases}$$

For a given ordinal α take a point x_α, whose center has depth α (such a point exists by Theorem 3.11), and denote $W = \omega_f(x_\alpha) \cup \{f^n(x_\alpha)\}_{n=0}^{\infty}$. Let $g \in \mathcal{C}(I^2, I^2)$ be such that, for any $(x, y) \in I^2$, we have $g(x, y) = y$ if $y = 1$ or if $x \in W$ and $y = 0$, and $g(x, y) > y$ otherwise. Finally, put $F(x, y) = (f(x), g(x, y))$. Then it can be easily shown that the depth of the center of F is α.

To prove Theorem A we will proceed as follows. Note that Statements (a)–(c) imply positive topological entropy because any map with zero topological entropy cannot have two different perfect ω-limit sets $\omega_f(x_1) \subset \omega_f(x_2)$ (cf., e.g. [Sharkovskii, 1968; Sharkovskii *et al.*, 1993] or [Bruckner & Smítal, 1993]). To prove that (a)–(c) of Theorem A are necessary for positive topological entropy we perform several steps. In Sec. 3, we prove that an analogous result holds for the full one-sided shift in the space of sequences of two symbols. Afterwards, in Sec. 4, we extend this result first to piecewise monotone maps on the interval and, finally, to arbitrary maps in $\mathcal{C}(I, I)$. Section 2 contains a few auxiliary results.

2. Auxiliary Results

This section is devoted to a few technical auxiliary results of various nature. We begin with the following elementary lemma, whose proof we omit.

Lemma 2.1. *Let X and Y be compact metric spaces. Assume that $g : Y \longrightarrow Y$, $\tau : X \longrightarrow X$ and $\varphi : Y \longrightarrow X$ are continuous maps such that $\varphi \circ g = \tau \circ \varphi$. Then, for every $x \in Y$, $\omega_\tau(\varphi(x)) = \varphi(\omega_g(x))$.*

Corollary 2.2. *With the notation from the preceding lemma, for each $x \in M$, the following statements hold:*

(a) $W_\tau^\alpha(\varphi(x)) = \varphi(W_g^\alpha(x))$, *for any ordinal* α.
(b) $d_\tau(\varphi(x)) \leq d_g(x)$.
(c) *If φ is a homeomorphism then* $d_\tau(\varphi(x)) = d_g(x)$.

Proof. Statement (a) can be proved by transfinite induction and Lemma 2.1. Statements (b) and (c) are direct consequences of Lemma 2.1 and (a). ■

Proposition 2.3. *Let f be a continuous self-map of a compact metric space Y such that for some $r \in \mathbb{N}$ and $Y^* \subset Y$, $Y = \cup_{n=0}^{r-1} f^n(Y^*)$ and $f^i(Y^*) \cap f^j(Y^*) = \emptyset$ whenever $0 \leq i < j < r$. Then, for each $y \in Y$ and each ordinal α, $W_f^\alpha(y) = \cup_{k=0}^{r-1} W_{f^r}^\alpha(f^k(y))$ and $d_{f^r}(y) = d_f(y)$.*

Proof. We note that for each $z \in Y$ and each ordinal γ, by Corollary 2.2,

(i) $f^l(W_{f^r}^\gamma(z)) = W_{f^r}^\gamma(f^l(z))$ for each $l \in \mathbb{N}$, and
(ii) $W_{f^r}^\gamma(f^i(z)) \cap W_{f^r}^\gamma(f^j(z)) = \emptyset$ whenever $0 \leq i < j < r$ (this can be easily deduced from the assumption $f^i(Y^*) \cap f^j(Y^*) = \emptyset$).

The first statement of the proposition can be proved by induction on α. An elementary proof for $\alpha = 0$ can be found, e.g. in [Block & Coppel, 1992]. Now the first statement for $\alpha > 0$ and the second one follow from (i) and (ii). ■

We conclude the section stating and proving a simple lemma about the computation of the depth of a point from the depth of another point.

Lemma 2.4. *Let $f : X \longrightarrow X$, let $x, u \in X$ and let α be a countable ordinal. Assume that $W_f^\alpha(u) = \omega_f(x)$ and either $d_f(u) \geq \alpha$ or $d_f(x) > 0$. Then $d_f(u) = \alpha + d_f(x)$.*

Proof. Let β be a countable ordinal. By transfinite induction it can be proved easily that $W_f^{\alpha+\beta}(u) = W_f^{\beta}(x)$. Therefore, for each countable ordinal $\beta \geq d_f(x)$,

$$W_f^{\alpha+\beta}(u) = W_f^{\beta}(x) = W_f^{d_f(x)}(x) = C_\tau(x) .$$

So, $d_f(u) \leq \alpha + d_f(x)$. If $d_f(x) = 0$ then the lemma holds because $d_f(u) \geq \alpha$. If $d_f(x) \geq 1$ then for each ordinal $0 \leq \beta < d_f(x)$ we have

$$W_f^{\alpha+\beta}(u) = W_f^{\beta}(x) \neq W_f^{d_f(x)}(x) = W_f^{\alpha+d_f(x)}(u)$$

and, hence, $d_f(u) \geq \alpha + d_f(x)$. This completes the proof of the lemma. ∎

3. Symbolic Dynamical Systems

In this section we prove that Theorem A holds with I and f replaced by the standard one-sided shift with two elements (see Theorem 3.11).

In the rest of this section X denotes the space $\{0, 1\}^{\mathbb{N}}$ endowed with the product topology. Hence, X is compact. Let τ be the shift map from X to itself, i.e. $\tau(\underline{\mathbf{x}}) = (x_2, x_3, \ldots)$, for any $\underline{\mathbf{x}} = (x_1, x_2, \ldots) \in X$.

As usual, the concatenation of any two finite sequences $\underline{x}$ and $\underline{y}$ of zeroes and ones is denoted by $\underline{xy}$. For $n \in \mathbb{Z}^+ \cup \{\infty\}$ the concatenation of $\underline{x}$ with itself n times is denoted by $\underline{x}^n$. In particular, 0^0 and 1^0 denote the empty sequence. If $\underline{x}$ and $\underline{y}$ are sequences (finite or infinite) we say that $\underline{x}$ is *contained* in $\underline{y}$ provided $\underline{y} = \underline{z_1 x z_2}$, where $\underline{z_1}$ and $\underline{z_2}$ can be empty. Also, if the first l symbols of $\underline{x}$ coincide with the first l symbols of $\underline{y}$ we write $\underline{x} =_l \underline{y}$. Lastly, we denote by $|\underline{x}| \in \mathbb{Z}^+ \cup \{\infty\}$ the length of a sequence $\underline{x}$.

Remark 3.1. Let $\{\underline{\mathbf{y_n}}\}_{n\in\mathbb{N}}$ be a sequence in X and let $\underline{\mathbf{y}} \in X$. Since X is endowed with the product topology, $\lim_{n\to\infty} \underline{\mathbf{y_n}} = \underline{\mathbf{y}}$ if and only if for each $l \in \mathbb{N}$ there exists n_l such that $\underline{\mathbf{y_n}} =_l \underline{\mathbf{y}}$ for each $n \geq n_l$.

For each subset M of $\mathbb{N}$ we denote by $X(M)$ the subset of X consisting of all sequences of the form

$$1^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots 0^{n_{k-1}}1^{m_k}0^{\infty} ,$$

$$1^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots 1^{m_k}0^{n_k}1^{\infty} ,$$

and

$$1^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots 1^{m_i}0^{n_i}\cdots ,$$

where $k, m_1 \in \mathbb{Z}^+$, $n_i \in \mathbb{N}$ for every i, $m_1 \leq \sup M$ and $m_i \in M$ for every $i > 1$. Note that $0^\infty, 1^\infty \in X(M)$ for each M.

For $m \in \mathbb{Z}^+ \cup \{\infty\}$ set $E_m = \{1^i 0^\infty, 1^i 0^j 1^\infty : i, j \geq 0, i \leq m\}$. We note that $E_\infty = \bigcup_{m=0}^{\infty} E_m$ and $E_m \subset X(M)$ whenever $m \leq \sup M$.

Let $M \subset \mathbb{N}$ and let $\underline{\mathbf{x}} \in X(M)$. Set

$$A(\underline{\mathbf{x}}) = \{n \in \mathbb{N} : 1^n 0^n \text{ is contained in } \underline{\mathbf{x}}\} .$$

Note that, for each $\underline{\mathbf{x}} \in X(M)$, $A(\underline{\mathbf{x}})$ is either finite or the set of all positive integers. When $A(\underline{\mathbf{x}}) = \mathbb{N}$ we will say that $\underline{\mathbf{x}}$ is *an essential point of* $X(M)$. Moreover, $X(M)$ has an essential point if and only if M is infinite.

The following lemma summarizes some of the properties of the spaces $X(M)$.

Lemma 3.2. *Let M, $M' \subset \mathbb{N}$. Then*

(a) *$X(M)$ is compact and τ-invariant. Moreover, if $M \neq \mathbb{N}$ then it is also nowhere dense.*
(b) *If $M \subset M'$ then $X(M) \subset X(M')$.*
(c) *If M is infinite then $X(M)$ is perfect and there is an essential point $\underline{\mathbf{x_M}}$ of $X(M)$ such that $\omega_\tau(\underline{\mathbf{x_M}}) = X(M)$.*
(d) *If $M \cap M' = \emptyset$ then $X(M) \cap X(M') = E_m$ with $m = \min\{\sup M, \sup M'\}$.*
(e) *If $M \cap M'$ is infinite then $X(M) \cap X(M') = X(M \cap M')$.*

Proof. From the definitions it follows that $X(M)$ is τ-invariant and that statements (b), (d) and (e) hold. By Remark 3.1, $X(M)$ is closed and, hence, compact. If $M \neq \mathbb{N}$ and $\underline{\mathbf{x}} \in X(M)$, then no neighborhood of $\underline{\mathbf{x}}$ is contained in $X(M)$. Consequently, $X(M)$ is nowhere dense and, hence, the proof of (a) is complete. Now assume that M is infinite and let S be the set of all finite sequences of the form

$$1^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots 1^{m_k}0^{n_k} ,$$

where $k, n_i \in \mathbb{N}$ and $m_i \in M$. Let $\{\underline{x_n}\}_{n=1}^{\infty}$ be an enumeration of S and

$$\underline{\mathbf{x_M}} = \underline{x_1 x_2 x_3} \cdots .$$

Clearly, $\underline{\mathbf{x_M}}$ is an essential point in $X(M)$ and $\omega_\tau(\underline{\mathbf{x_M}}) = X(M)$. So, (c) holds. ∎

Let $\varphi : M \longrightarrow M'$ be a bijection between two *infinite* subsets of $\mathbb{N}$. Then we denote by Φ the map from $X(M)$ to $X(M')$ such that each point from $X(M)$ is mapped to a point having the same representation (in terms of blocks of zeroes and ones) but with m_i replaced by $\varphi(m_i)$ for each $i > 1$ (here we use the notation from the definition of the sets $X(M)$).

Remark. When M and M' are finite Φ need not map $X(M)$ into $X(M')$ since $m_1 \leq \sup M$ need not imply $m_1 \leq \sup M'$.

Lemma 3.3. *The map Φ is a homeomorphism and $\tau \circ \Phi = \Phi \circ \tau$.*

Proof. Clearly, Φ is a bijection and $\tau \circ \Phi = \Phi \circ \tau$. Since X is a second countable space and Φ is a bijection it suffices to show that, for each convergent sequence $\{\underline{\mathbf{x_n}}\}_{n\in\mathbb{N}}$ in $X(M)$,

$$\lim_{n\to\infty} \Phi(\underline{\mathbf{x_n}}) = \Phi(\lim_{n\to\infty} \underline{\mathbf{x_n}}) .$$

This is a consequence of Remark 3.1 and so the proof is complete. ■

The next results are devoted to the construction of points having a prescribed center and depth (see Proposition 3.10).

For each $\underline{\mathbf{x}} \in X$ and $n \in \mathbb{N}$, we denote by $[\underline{\mathbf{x}}]_n$ the sequence formed by the first n symbols of $\underline{\mathbf{x}}$.

Lemma 3.4. *Let $K \subset \mathbb{N}$, let $\underline{\mathbf{x}}$ be an essential point of $X(K)$ and let $\{m_i\}_{i\in\mathbb{N}}$ be an increasing sequence of positive integers. Let $\{n_i\}_{i\in\mathbb{N}}$ be a sequence of positive integers such that $n_i \geq m_i$ for each $i \in \mathbb{N}$. Then there exist finite sequences $\underline{x_1}$, $\underline{x_2}, \ldots$ such that*

$$\underline{\mathbf{x}} = \underline{x_1}1^{n_1}0\underline{x_2}1^{n_2}0^2 \cdots \underline{x_k}1^{n_k}0^k \cdots .$$

Moreover, if we set

$$\underline{\mathbf{u}} = \underline{x_1}1^{n_1}01^{m_1}0\underline{x_2}1^{n_2}0^21^{m_2}0^2 \cdots \underline{x_k}1^{n_k}0^k1^{m_k}0^k\underline{x_{k+1}} \cdots ,$$

then $\underline{\mathbf{u}}$ is an essential point of $X(K \cup \{m_i\}_{i\in\mathbb{N}})$ and

$$\omega_\tau(\underline{\mathbf{u}}) = \omega_\tau(\underline{\mathbf{x}}). \tag{1}$$

Proof. The existence of the sequences $\underline{x_1}$, $\underline{x_2}, \ldots$ is a direct consequence of the fact that $\underline{\mathbf{x}}$ is essential. Clearly, $\underline{\mathbf{u}} \in X(K \cup \{m_i\}_{i\in\mathbb{N}})$ and, since $\underline{\mathbf{x}}$ is essential, $\underline{\mathbf{u}}$ is also essential. Now, let us prove (1). We start by showing that $\omega_\tau(\underline{\mathbf{u}}) \supset \omega_\tau(\underline{\mathbf{x}})$. Take $\underline{\mathbf{z}} \in \omega_\tau(\underline{\mathbf{x}})$. By Remark 3.1, for each $l \in \mathbb{N}$, there exists an unbounded set $S_l \subset \mathbb{N}$ such that for each $i \in S_l$, $[\underline{\mathbf{z}}]_l$ is contained in $0^l\underline{x_i}1^{n_i}0^l$. Therefore, there exists a strictly increasing sequence $\{j'_l\}_{l=1}^\infty$ such that for each $l \in \mathbb{N}$, $\tau^{j'_l}(\underline{\mathbf{u}})$ starts with $0^l\underline{x_i}1^{n_i}0^l$, where $i \in S_l$. Thus, there exists a sequence $\{j_l\}_{l=1}^\infty$ such that $j'_l \leq j_l$ and $\tau^{j_l}(\underline{\mathbf{u}}) =_l \underline{\mathbf{z}}$ for each $l \in \mathbb{N}$. Consequently, $\underline{\mathbf{z}} \in \omega_\tau(\underline{\mathbf{u}})$.

Now we prove the other inclusion. Let $\underline{\mathbf{z}} \in \omega_\tau(\underline{\mathbf{u}})$ and let T be the set of all $l \in \mathbb{N}$ such that $[\underline{\mathbf{z}}]_l$ is contained in $0^l\underline{x_i}1^{n_i}0^l$ for some i. If T is infinite then one can construct a sequence $\{k_l\}_{l=1}^\infty$ tending to infinity, such that $\tau^{k_l}(\underline{\mathbf{x}}) =_l \underline{\mathbf{z}}$ for each $l \in \mathbb{N}$. Consequently, $\underline{\mathbf{z}} \in \omega_\tau(\underline{\mathbf{x}})$. If T is finite then, by Remark 3.1, there exists $L \in \mathbb{N}$ such that $[\underline{\mathbf{z}}]_l$ is contained in $0^l1^{m_i}0^l$ for each i, $l > L$. Hence, either $\underline{\mathbf{z}} \in \{0^i1^\infty\}_{i\in\mathbb{Z}^+}$ or $\underline{\mathbf{z}} \in \{1^i0^\infty\}_{i\in\mathbb{Z}^+}$. Since $\underline{\mathbf{x}}$ is essential, $\{0^i1^\infty\}_{i\in\mathbb{Z}^+} \cup \{1^i0^\infty\}_{i\in\mathbb{Z}^+} \subset \omega_\tau(\underline{\mathbf{x}})$. ■

Lemma 3.5. *Let $M \subset \mathbb{N}$ and let $\underline{\mathbf{u}}$ be an essential point of $X(M)$. For any $n \in \mathbb{N}$, let $\underline{u_n}$ be a finite sequence ending with 1^n0^n such that $\underline{\mathbf{u}}$ begins with $\underline{u_n}$. Let*

$$\underline{\mathbf{y}} = \underline{u_1}\underline{u_2} \cdots \underline{u_n} \cdots .$$

Then

$$\omega_\tau(\underline{\mathbf{y}}) = \{0^i\underline{\mathbf{u}}\}_{i\in\mathbb{N}} \cup \{\tau^i(\underline{\mathbf{u}})\}_{i\in\mathbb{Z}^+} \cup \omega_\tau(\underline{\mathbf{u}}) \tag{2}$$

Proof. For each $n \in \mathbb{N}$, define $\underline{\mathbf{y_n}} = \underline{u_n}\underline{u_{n+1}} \cdots$ and fix $k \in \mathbb{Z}^+$. Note that for each $n > k$ there exists $l_n \in \mathbb{N}$ such that $0^k\underline{\mathbf{y_n}} = \tau^{l_n}(\underline{\mathbf{y}})$. Moreover, we can assume that $l_n < l_{n+1}$. So, by Remark 3.1, $\lim_{n\to\infty} 0^k\underline{\mathbf{y_n}} = 0^k\underline{\mathbf{u}}$. Hence, $0^k\underline{\mathbf{u}} \in \omega_\tau(\underline{\mathbf{y}})$ and thus,

$$\{0^i\underline{\mathbf{u}}\}_{i\in\mathbb{N}} \cup \{\tau^i(\underline{\mathbf{u}})\}_{i\in\mathbb{Z}^+} \cup \omega_\tau(\underline{\mathbf{u}}) \subset \omega_\tau(\underline{\mathbf{y}}) .$$

To prove that the other inclusion holds, fix a $\underline{\mathbf{z}} \in \omega_\tau(\underline{\mathbf{y}})$. By Remark 3.1, there exists a sequence $\{l_n\}_{n=0}^\infty$ tending to infinity such that $[\underline{\mathbf{z}}]_n$ is contained in $0^n\underline{u_{l_n}}$. So, $[\underline{\mathbf{z}}]_n$ is contained in $0^n\underline{u_k}$ for any $k \geq l_n$. Thus, for each $n \in \mathbb{N}$ and $k \geq l_n$, either $[\underline{\mathbf{z}}]_n$ is contained in $\underline{u_k}$ or $[\underline{\mathbf{z}}]_n =_n 0^i\underline{u_k}$, where $0 < i \leq n$. In any case, $z \in \{0^i\underline{\mathbf{u}} : i > 0\} \cup \{\tau^i(\underline{\mathbf{u}}) : i \geq 0\} \cup \omega_\tau(\underline{\mathbf{u}})$. This completes the proof of the lemma. ■

Lemma 3.6. *Let $K \subset \mathbb{N}$, let $\underline{\mathbf{x}}$ be an essential point of $X(K)$ and let $\{m_i\}_{i\in\mathbb{N}}$ be an increasing sequence in $\mathbb{N}\backslash K$. Then there is an essential point*

$\underline{\mathbf{y}} \in X(K \cup \{m_i\}_{i\in\mathbb{N}})$ such that $\omega_\tau(\underline{\mathbf{y}}) \cap X(K) = \omega_\tau(\underline{\mathbf{x}}) = W_\tau^1(\underline{\mathbf{y}})$ and $d_\tau(\underline{\mathbf{y}}) = 1 + d_\tau(\underline{\mathbf{x}})$.

Proof. By Lemma 3.4 there exists an essential point $\underline{\mathbf{u}} \in X(K \cup \{m_i\}_{i\in\mathbb{N}})$ such that $\omega_\tau(\underline{\mathbf{u}}) = \omega_\tau(\underline{\mathbf{x}}) \subset X(K)$. Now let $\underline{\mathbf{y}} \in X(K \cup \{m_i\}_{i\in\mathbb{N}})$ be the point given by Lemma 3.5. By (2) and (1),

$$\omega_\tau(\underline{\mathbf{y}}) = \{0^i\underline{\mathbf{u}}\}_{i\in\mathbb{N}} \cup \{\tau^i(\underline{\mathbf{u}})\}_{i\in\mathbb{Z}^+} \cup \omega_\tau(\underline{\mathbf{x}})$$

Since $\underline{\mathbf{u}}$ contains all sequences of the form $01^{m_i}0$ with $m_i \notin K$, $\omega_\tau(\underline{\mathbf{y}}) \cap X(K) = \omega_\tau(\underline{\mathbf{x}})$. On the other hand, by (1),

$$W_\tau^1(\underline{\mathbf{y}}) = \omega_\tau(\underline{\mathbf{u}}) \cup W_\tau^1(\underline{\mathbf{x}}) = \omega_\tau(\underline{\mathbf{x}}) \neq W_\tau^0(\underline{\mathbf{y}})\,.$$

Hence, by Lemma 2.4, $d_\tau(\underline{\mathbf{y}}) = 1 + d_\tau(\underline{\mathbf{u}})$. ■

Lemma 3.7. *Let α be a countable ordinal. Assume that M and K are disjoint infinite sets of positive integers. Let $\underline{\mathbf{u}} \in X(K)$ and, for each $\beta < \alpha$, let $\underline{\mathbf{x}_\beta} \in X(M \cup K)$ be such that*

$$\omega_\tau(\underline{\mathbf{x}_\beta}) \cap X(K) = \omega_\tau(\underline{\mathbf{u}})\,, \tag{3}$$

$$W_\tau^\beta(\underline{\mathbf{x}_\beta}) = \omega_\tau(\underline{\mathbf{u}}) \text{ and } d_\tau(\underline{\mathbf{x}_\beta}) = \beta + d_\tau(\underline{\mathbf{u}})\,. \tag{4}$$

Then, for each $\beta < \alpha$, there is a point $\underline{\mathbf{u}_\beta} \in X(M \cup K)$ such that

$$\omega_\tau(\underline{\mathbf{u}_\beta}) \cap X(K) = \omega_\tau(\underline{\mathbf{u}})\,, \tag{5}$$

$$\omega_\tau(\underline{\mathbf{u}_\beta}) \cap \omega_\tau(\underline{\mathbf{u}_{\beta'}}) = \omega_\tau(\underline{\mathbf{u}}) \tag{6}$$

for $\beta' < \alpha$ and $\beta \neq \beta'$,

$$W_\tau^\beta(\underline{\mathbf{u}_\beta}) = \omega_\tau(\underline{\mathbf{u}}) \text{ and } d_\tau(\underline{\mathbf{u}_\beta}) = \beta + d_\tau(\underline{\mathbf{u}})\,. \tag{7}$$

Proof. Let $\{M_\beta\}_{\beta<\alpha}$ be a transfinite sequence of pairwise disjoint infinite subsets of M. For each ordinal $\beta < \alpha$, let $\varphi_\beta : M \cup K \longrightarrow M_\beta \cup K$ be a bijection such that $\varphi_\beta|_K$ is the identity and let $\Phi_\beta : X(M \cup K) \longrightarrow X(M_\beta \cup K)$ be the associated homeomorphism (cf. Lemma 3.3). Note that $\Phi_\beta|_{X(K)}$ is the identity. For $\beta < \alpha$, define $\underline{\mathbf{u}_\beta} = \Phi_\beta(\underline{\mathbf{x}_\beta})$. Then, (5) is a consequence of (3) and Lemma 2.1. Since $\omega_\tau(\underline{\mathbf{u}_\beta}) \subset X(M_\beta \cup K)$, (6) follows from Lemma 3.2(e) and (5). Finally, since Φ_β is a homeomorphism, (7) follows from (4) and Corollary 2.2(a) and (c). ■

Lemma 3.8. *Let α be a countable limit ordinal and let M and K be disjoint infinite sets of positive integers. Let $\underline{\mathbf{u}}$ be an essential point of $X(K)$ and, for each $\beta < \alpha$, let $\underline{\mathbf{u}_\beta}$ be an essential point of $X(M \cup K)$ such that (5) and (7) hold. Then there is an essential point $\underline{\mathbf{z}_\alpha}$ in $X(M \cup K)$ such that $\omega_\tau(\underline{\mathbf{z}_\alpha}) \cap X(K) = \omega_\tau(\mathbf{u}) = W_\tau^\alpha(\underline{\mathbf{z}_\alpha})$ and $d_\tau(\underline{\mathbf{z}_\alpha}) = \alpha + d_\tau(\underline{\mathbf{u}})$.*

Proof. By Lemma 3.7 we may also assume that (6) is satisfied for each distinct $\beta, \beta' < \alpha$. For any $0 < \beta < \alpha$ and any $n \in \mathbb{N}$, let $\underline{u_\beta^n}$ be a (finite) sequence ending with n consecutive zeroes such that $\underline{\mathbf{u}_\beta}$ begins with $\underline{u_\beta^n}$. Let $\{\beta(k)\}_{k=1}^\infty$ be a sequence of ordinals smaller than α containing any ordinal $0 < \beta < \alpha$ infinitely many times. Define

$$\underline{\mathbf{z}_\alpha} = \underline{u_{\beta(1)}^1 u_{\beta(2)}^2 u_{\beta(3)}^3} \cdots \underline{u_{\beta(n)}^n} \cdots\,.$$

Clearly, $\underline{\mathbf{z}_\alpha}$ is an essential point of $X(M \cup K)$. Then, by an argument similar to that used in the proof of Lemma 3.5, we obtain

$$\begin{aligned} W_\tau^0(\underline{\mathbf{z}_\alpha}) &= \omega_\tau(\underline{\mathbf{z}_\alpha}) \\ &= \bigcup_{0<\beta<\alpha} \omega_\tau(\underline{\mathbf{u}_\beta}) \\ &\quad \cup \bigcup_{0<\beta<\alpha} \{0^k\underline{\mathbf{u}_\beta} : k > 0\} \\ &\quad \cup \bigcup_{0<\beta<\alpha} \{\tau^k(\underline{\mathbf{u}_\beta}) : k \geq 0\}\,. \end{aligned} \tag{8}$$

Hence,

$$W_\tau^1(\underline{\mathbf{z}_\alpha}) = \bigcup_{0<\beta<\alpha} \omega_\tau(\underline{\mathbf{u}_\beta}) = \bigcup_{0<\beta<\alpha} W_\tau^0(\underline{\mathbf{u}_\beta})$$

and, by induction on n,

$$W_\tau^{n+1}(\underline{\mathbf{z}_\alpha}) = \bigcup_{0<\beta<\alpha} W_\tau^n(\underline{\mathbf{u}_\beta}) \tag{9}$$

for any $n \in \mathbb{Z}^+$. Denote by ω the first infinite ordinal. By using the definition of $W_\tau^\omega(\cdot)$, (9), (6), (7) and again the definition of $W_\tau^\omega(\cdot)$ we obtain

$$\begin{aligned} W_\tau^\omega(\underline{\mathbf{z}_\alpha}) &= \bigcap_{n=0}^\infty W_\tau^{n+1}(\underline{\mathbf{z}_\alpha}) \\ &= \bigcap_{n=0}^\infty \left(\bigcup_{0<\beta<\alpha} W_\tau^n(\underline{\mathbf{u}_\beta})\right) \\ &= \bigcup_{0<\beta<\alpha} \left(\bigcap_{n=0}^\infty W_\tau^n(\underline{\mathbf{u}_\beta})\right) \\ &= \bigcup_{0<\beta<\alpha} W_\tau^\omega(\underline{\mathbf{u}_\beta})\,. \end{aligned} \tag{10}$$

By (6), (7), (10) and transfinite induction beginning with $\gamma = \omega$, we obtain

$$W_\tau^\gamma(\underline{\mathbf{z}_\alpha}) = \bigcup_{0<\beta<\alpha} W_\tau^\gamma(\underline{\mathbf{u}_\beta}) \tag{11}$$

whenever $\omega \le \gamma \le \alpha$. By (7), $W_\tau^\gamma(\underline{\mathbf{u}_\beta}) \supset \omega_\tau(\underline{\mathbf{u}})$ for $\gamma \le \beta < \alpha$. Hence, by (11), $W_\tau^\gamma(\underline{\mathbf{z}_\alpha}) \supset \omega_\tau(\underline{\mathbf{u}})$ and, consequently, $W_\tau^\alpha(\underline{\mathbf{z}_\alpha}) \supset \omega_\tau(\underline{\mathbf{u}})$. On the other hand, by (7), $W_\tau^\gamma(\underline{\mathbf{u}_\beta}) \subset \omega_\tau(\underline{\mathbf{u}})$ whenever $0 < \beta \le \gamma < \alpha$. Therefore, (11) and (7) yield

$$W_\tau^\gamma(\underline{\mathbf{z}_\alpha}) = \bigcup_{\gamma\le\beta<\alpha} W_\tau^\gamma(\underline{\mathbf{u}_\beta}) \subset \bigcup_{\gamma\le\beta<\alpha} \omega_\tau(\underline{\mathbf{u}_\beta}) . \tag{12}$$

Thus, by (6),

$$W_\tau^\alpha(\underline{\mathbf{z}_\alpha}) \subset \bigcap_{\gamma<\alpha}\left(\bigcup_{\gamma\le\beta<\alpha} \omega_\tau(\underline{\mathbf{u}_\beta})\right) = \omega_\tau(\underline{\mathbf{u}})$$

and, hence, $W_\tau^\alpha(\underline{\mathbf{z}_\alpha}) = \omega_\tau(\underline{\mathbf{u}})$.

Now we claim that $d_\tau(\underline{\mathbf{z}_\alpha}) \ge \alpha$. To prove this it suffices to show that $W_\tau^\gamma(\underline{\mathbf{z}_\alpha}) \ne W_\tau^\alpha(\underline{\mathbf{z}_\alpha})$ for each $\gamma < \alpha$. For each $\gamma < \alpha$, choose β_γ such that $\gamma < \beta_\gamma < \alpha$. Then, by (7), $W_\tau^\gamma(\underline{\mathbf{u}_{\beta_\gamma}}) \supsetneq W_\tau^{\beta_\gamma}(\underline{\mathbf{u}_{\beta_\gamma}}) = \omega_\tau(\underline{\mathbf{u}})$. Consequently, if $\alpha > \omega$, by (12) we have

$$\begin{aligned} W_\tau^\gamma(\underline{\mathbf{z}_\alpha}) &= \bigcup_{\gamma\le\beta<\omega} W_\tau^\gamma(\underline{\mathbf{u}_\beta}) \supset W_\tau^\gamma(\underline{\mathbf{u}_{\beta_\gamma}}) \\ &\supsetneq \omega_\tau(\underline{\mathbf{u}}) = W_\tau^\alpha(\underline{\mathbf{z}_\alpha}) . \end{aligned}$$

If $\alpha = \omega$ then the claim follows in a similar way by (9) and (7). Therefore, since $W_\tau^\alpha(\underline{\mathbf{z}_\alpha}) = \omega_\tau(\underline{\mathbf{u}})$ and $d_\tau(\underline{\mathbf{z}_\alpha}) \ge \alpha$, by Lemma 2.4 we obtain $d_\tau(\underline{\mathbf{z}_\alpha}) = \alpha + d_\tau(\underline{\mathbf{u}})$.

Now, we prove that $\omega_\tau(\underline{z_\alpha}) \cap X(K) = \omega_\tau(\underline{\mathbf{u}})$. In view of (5), (8) and Lemma 3.2(a), it is enough to show that

$$\{0^k\underline{\mathbf{u}_\beta} : k > 0\} \cap X(K) =$$
$$\{\tau^k(\underline{\mathbf{u}_\beta}) : k \ge 0\} \cap X(K) = \emptyset$$

for any $k \in \mathbb{Z}^+$ and $0 < \beta < \alpha$. Otherwise, $\tau^k(\underline{\mathbf{u}_\beta}) \in X(K)$ or $0^k\underline{\mathbf{u}_\beta} \in X(K)$ which imply $\omega_\tau(\underline{\mathbf{u}_\beta}) \subset X(K)$ and, by (5), $W_\tau^0(\underline{\mathbf{u}_\beta}) = \omega_\tau(\underline{\mathbf{u}_\beta}) = \omega_\tau(\underline{\mathbf{u}})$ which contradicts (7). This ends the proof of the lemma. ■

Lamma 3.9. *Let M and K be disjoint infinite sets of positive integers, let α be a countable ordinal and let $\underline{\mathbf{x_K}}$ be an essential point of $X(K)$. Then there is an essential point $\underline{\mathbf{z}_\alpha}$ from $X(M \cup K)$ such that*

$$\omega_\tau(\underline{\mathbf{z}_\alpha}) \cap X(K) = \omega_\tau(\underline{\mathbf{x_K}}) = W_\tau^\alpha(\underline{\mathbf{z}_\alpha})$$

and $d_\tau(\underline{\mathbf{z}_\alpha}) = \alpha + d_\tau(\underline{\mathbf{x_K}})$.

Proof. We prove the proposition by transfinite induction. If $\alpha = 0$, the lemma holds by taking $\underline{\mathbf{z_0}} = \underline{\mathbf{x_K}}$. Now fix a countable ordinal α and assume that for any $\beta < \alpha$, any pair of disjoint infinite sets of positive integers M' and K' and any essential point $\underline{\mathbf{x_{K'}}} \in X(K')$, there exists $\underline{\mathbf{z}_\beta(\mathbf{x_{K'}})}$ such that the lemma holds with $\underline{\mathbf{x_K}}$, and α and $\underline{\mathbf{z}_\alpha}$ replaced by $\underline{\mathbf{x_{K'}}}$, β and $\underline{\mathbf{z}_\beta(\mathbf{x_{K'}})}$, respectively.

Assume now that $\alpha = \beta + 1$ for some countable ordinal β. Let M' be a proper infinite subset of M such that $M \setminus M'$ is infinite and let $\{m_i\}_{i\in\mathbb{N}}$ be an increasing sequence in $M \setminus M'$. Then, let $\underline{\mathbf{y}}$ be the essential point from $X(K \cup \{m_i\}_{i\in\mathbb{N}})$ given by Lemma 3.6 with $\underline{\mathbf{x}}$ replaced by $\underline{\mathbf{x_K}}$. By the induction hypothesis, there exists $\underline{\mathbf{z}_\beta(\mathbf{y})} \in X(M' \cup K \cup \{m_i\}_{i\in\mathbb{N}})$ such that

$$\begin{aligned} \omega_\tau(\underline{\mathbf{z}_\beta(y)}) \cap X(K \cup \{m_i\}_{i\in\mathbb{N}}) &= \omega_\tau(\underline{\mathbf{y}}) \\ &= W_\tau^\beta(\underline{\mathbf{z}_\beta(y)}) \end{aligned}$$

and $d_\tau(\underline{\mathbf{z}_\beta(\mathbf{y})}) = \beta + d_\tau(\underline{\mathbf{y}})$. Put $\underline{\mathbf{z}_\alpha} = \underline{\mathbf{z}_\beta(\mathbf{y})}$. Then, by Lemmas 3.6 and 3.2(b),

$$\omega_\tau(\mathbf{z}_\alpha) \cap X(K) = \omega_\tau(\mathbf{y}) \cap X(K) = \omega_\tau(\underline{\mathbf{x_K}})$$

and

$$\begin{aligned} d_\tau(\underline{\mathbf{z}_\alpha}) &= d_\tau(\underline{\mathbf{z}_\beta(y)}) = \beta + d_\tau(\underline{\mathbf{y}}) \\ &= \beta + 1 + d_\tau(\underline{\mathbf{x_K}}) = \alpha + d_\tau(\underline{\mathbf{x_K}}) . \end{aligned}$$

Moreover,

$$\begin{aligned} W_\tau^\alpha(\underline{\mathbf{z}_\alpha}) &= W_\tau^{\beta+1}(\underline{z_\alpha}) = \bigcup_{\underline{\mathbf{t}}\in W_\tau^\beta(\underline{\mathbf{z}_\beta(\mathbf{y})})} \omega_\tau(\underline{\mathbf{t}}) \\ &= \bigcup_{\underline{\mathbf{t}}\in\omega_\tau(\underline{y})} \omega_\tau(\underline{t}) = W_\tau^1(\underline{\mathbf{y}}) = \omega_\tau(\underline{\mathbf{x_K}}) . \end{aligned}$$

If α is a countable limit ordinal, then the statement follows immediately from Lemma 3.8 and the induction hypotheses with $M' = M$, $K' = K$ and $\underline{\mathbf{x_{K'}}} = \underline{\mathbf{x_K}}$. ■

Proposition 3.10. *Let M and K be disjoint infinite sets of positive integers. Then, for any* *countable ordinal α there exists $\underline{\mathbf{x}_\alpha} \in X(M \cup K)$ such that $C_\tau(\underline{\mathbf{x}_\alpha}) = X(K)$ and $d_\tau(\underline{\mathbf{x}_\alpha}) = \alpha$.*

Proof. In view of Lemma 3.2(c) there exists an essential point $\underline{\mathbf{x}_\mathbf{K}}$ in $X(K)$ such that $\omega_\tau(\underline{\mathbf{x}_\mathbf{K}}) = X(K)$. Let z_α be the point given by Lemma 3.9. We note that $C_\tau(\underline{\mathbf{x}_\mathbf{K}}) = X(K)$ and $d_\tau(\underline{\mathbf{x}_\mathbf{K}}) = 0$. Hence, $d_\tau(z_\alpha) = \alpha$ and $\omega_\tau(\underline{\mathbf{z}_\alpha}) \cap X(K) = \omega_\tau(\underline{\mathbf{x}_\mathbf{K}}) = W_\tau^\alpha(\underline{\mathbf{z}_\alpha})$. So, $C_\tau(z_\alpha) = W_\tau^\alpha(\underline{\mathbf{z}_\alpha}) = \omega_\tau(\underline{\mathbf{x}_\mathbf{K}}) = X(K)$. ∎

Theorem 3.11. *There exists a family $\{X_t\}_{t\in[0,\,1]}$ of nowhere dense perfect subsets of X such that*

(a) *$X_t \subset X_s$ and $X_t \neq X_s$ if $t < s$.*
(b) *For any $t \in [0, 1]$ there exists $\underline{\mathbf{x}_\mathbf{t}} \in X_t$ such that $\omega_\tau(\underline{\mathbf{x}_\mathbf{t}}) = X_t$.*
(c) *For any $t < s$ in $[0, 1]$ and any countable ordinal α there exists $\underline{\mathbf{y}_\mathbf{s}} \in X_s$ such that $d_\tau(\underline{\mathbf{y}_\mathbf{s}}) = \alpha$ and $C_\tau(\underline{\mathbf{y}_\mathbf{s}}) = X_t$.*

Proof. Let Q be the set of rational numbers in $[0, 1]$, and let $\{M_t\}_{t\in Q}$ be a family of mutually disjoint infinite sets of positive integers. For $t \in [0, 1]$ define $X_t = X(\cup_{r<t} M_r)$. Then the theorem follows from Lemma 3.2(a)–(c) and Proposition 3.10. ∎

4. Proof of Theorem A

It is well known (see for instance Corollary 16 of Chapter II of [Block & Coppel, 1992]) that for any continuous interval map with positive topological entropy there exists a compact invariant set B such that an iterate of the map restricted to B is semiconjugate to a full shift with two elements (compare also with [Young, 1981, Theorem 2.4]). In order to prove Theorem A we need a stronger result: two iterates of B by the map either coincide or are pairwise disjoint. This is proved in Proposition 4.4. More precisely, this proposition shows that any continuous interval map with positive topological entropy contains as a subsystem a compact invariant subset Σ of a subshift of finite type $(\mathcal{S}, \sigma|_\mathcal{S})$ such that, for some $r \in \mathbb{N}$ and $\Sigma^* \subset \mathcal{S}$,

(a) $\Sigma = \bigcup_{i=0}^{r-1} \tau^i(\Sigma^*)$,
(b) the sets $\tau^i(\Sigma^*)$ and $\tau^j(\Sigma^*)$ are disjoint if and only if $i \not\equiv j$ modulo r, and
(c) $\tau^r|_{\Sigma^*}$ is conjugate to a full shift with two elements.

We start with some notions and preliminary results. A set of closed subintervals of I with pairwise disjoint interiors will be called a *basic system*. Let A be a basic system. Any bijective map from the set $\{1, 2, \ldots, \mathrm{Card} A\}$ to A will be called a *coding of A*. Consider a triple (A, κ, f) where A is a basic system, κ is a coding of A and $f \in \mathcal{C}(I, I)$. To such a triple we can associate a subshift of finite type of order 2, denoted by $\Lambda(A, \kappa, f)$, as follows (here, when speaking about subshifts of finite type, we will use the notation and definitions from [Denker *et al.*, 1976]). The alphabet of the subshift will be the set $\{1, 2, \ldots, \mathrm{Card}\, A\}$ and the transition matrix is defined to be the $(\mathrm{Card}\, A)\times(\mathrm{Card}\, A)$ matrix $T = (t_{ij})$ defined by $t_{ij} = 1$ if $f(\kappa(i)) \supset \kappa(j)$ and $t_{ij} = 0$ otherwise. Any $(n + 1)$-block $\underline{a} = (a_0, a_1, \ldots, a_n)$ occurring in such a subshift of finite type is called a *loop* if $a_n = a_0$. The number n is called the *length* of the loop and is denoted by $|\underline{a}|$. We will say that a loop is *elementary* if $a_i \neq a_j$ whenever $i \neq j$ for $0 \leq i, j < n$.

Let $\underline{\mathbf{a}} = (a_0, a_1, \ldots)$ be either a block occurring in $\Lambda(A, \kappa, f)$ or an element from $\Lambda(A, \kappa, f)$. Given a point $z \in I$ we say that it *f-follows* $\underline{\mathbf{a}}$ if $f^i(z) \in \kappa(a_i)$ for each $i \geq 0$. It is easy to see that, since f is continuous and I is compact there exists at least one point in I which f-follows $\underline{\mathbf{a}}$. This elementary fact is known as the *Itinerary Lemma*.

Let $f \in \mathcal{C}(I, I)$ and let P be a periodic orbit of f. We will denote by $\langle P\rangle$ the *convex hull of P* (that is, the smallest closed connected set containing P). The closure of each connected component of $\langle P\rangle\backslash P$ is called a *P-basic interval.* We say that the map f is *P-monotone* if it is monotone on the closure of each connected component of $I\backslash P$.

Given a map $f \in \mathcal{C}(I, I)$ and a periodic orbit P of f we define an auxiliary map f_P as follows. For each $x \in \langle P\rangle$ denote by x^- and x^+ the two points from P such that $\langle x^-, x^+\rangle$ is a P-basic interval, $x \in \langle x^-, x^+\rangle$ and $f(x^-) < f(x^+)$. (Here $\langle x^-, x^+\rangle$ stands for $\langle\{x^-, x^+\}\rangle$; note that $f(x^-) \neq f(x^+)$ since P is a periodic orbit.) Then we define f_P as the map from $\mathcal{C}(I, I)$ such that it is constant on each connected component of $I\backslash\langle P\rangle$ and $f_P(x) = \min\{f(x^+), \max_{z\in\langle x^-, x\rangle} f(z)\}$ for each $x \in \langle P\rangle$. Note that f_P is P-monotone and $f_P|_P = f|_P$.

Now we start to develop the tools to prove Theorem A. Let $f \in \mathcal{C}(I, I)$ be with positive topological entropy. By Theorem 4.4.10 and Corollary 4.4.7 of [Alsedà *et al.*, 1993], it follows that f has a periodic orbit P such that $h(f_P)$, the topological entropy of f_P, is positive. In the rest of this section we will assume that f and P are fixed.

Let A be the set of all P-basic intervals of f_P, let κ^* be a coding of A and let $M = (m_{ij})$ denote the transition matrix of $\Lambda(A, \kappa^*, f_P)$. By changing κ^* (if necessary) the matrix M can be written in block form as:

$$M = \begin{pmatrix} M_1 & 0 & 0 & 0 \\ * & M_2 & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & M_l \end{pmatrix},$$

where each block M_i is a square matrix which is either irreducible or zero [Gantmacher, 1959] (recall that a non-negative square matrix L is called *irreducible* if, for each entry of L, there exists $l \in \mathbb{N}$ such that the corresponding entry of L^l is positive).

Since $h(f_P) > 0$, from [Alsedà *et al.*, 1993, Theorem 4.4.5] it follows that the spectral radius of M is larger than one. So, there exists a block $T = M_i$ of M having spectral radius larger than one. Hence, T is an irreducible matrix which is not a permutation matrix. Let $U = \{j \in \mathbb{N} : m_{jj}$ belong to the block matrix $M_i\}$ and let κ be a coding of $\kappa^*(U)$. In what follows we will denote $\Lambda(\kappa^*(U), \kappa, f_P)$ by $\mathcal{S}$. Note that the transition matrix of $\mathcal{S}$ is precisely T.

The irreducibility of T implies that $\mathcal{S}$ has a loop $\underline{a}$. Without loss of generality we can assume that $\underline{a} = (a_0, a_1, \ldots, a_{t-1}, a_0)$ has minimal length. Clearly, it is elementary. Therefore, the fact that T is not a permutation matrix implies that there is a symbol k in the alphabet of $\mathcal{S}$ which does not lie in $\underline{a}$. Also, there is a loop from a_0 to a_0 through k. Thus, there is a loop in $\mathcal{S}$ containing a symbol from $\underline{a}$ and a symbol which does not lie in $\underline{a}$. Let $\underline{b}$ be a loop of minimal length having these properties. Denote by l a common symbol of $\underline{a}$ and $\underline{b}$. Without loss of generality we may assume that $\underline{a} = (a_0, a_1, \ldots, a_{|\underline{a}|-1}, a_0)$ and $\underline{b} = (b_0, b_1, \ldots, b_{|\underline{b}|-1}, b_0)$ with $a_0 = b_0 = l$ and $a_1 \neq b_1$. Note that, by the minimality of $\underline{a}$ and $\underline{b}$, l appears only at the beginning and at the end of $\underline{a}$ and $\underline{b}$, and $|\underline{b}| \geq |\underline{a}|$.

Let Σ^* denote the space of all elements from $\mathcal{S}$ which can be obtained by concatenating the loops $\underline{a}^3\underline{b}^2$ and $\underline{a}^2\underline{b}\underline{a}\underline{b}$ and let Σ denote $\cup_{n=0}^{\infty}\sigma^n(\Sigma^*)$. Set $r = |\underline{a}^3\underline{b}^2|$. Since $\sigma^r(\Sigma^*) = \Sigma^*$, we have $\Sigma = \cup_{n=0}^{r-1}\sigma^n(\Sigma^*)$. Note that both Σ^* and Σ are compact.

Our next objective is to show that Theorem 3.11 holds with X replaced by Σ and τ by $\sigma|_\Sigma$. To this end we consider the map $\psi : X \longrightarrow \Sigma^*$ defined in the following way. For each $\underline{\mathbf{x}} = \{x_i\}_{i=0}^{\infty} \in X$ we define $\psi(\underline{\mathbf{x}}) = d_0 d_1 \cdots \in \Sigma^*$ where

$$d_i = \begin{cases} \underline{a}^3\underline{b}^2 & \text{if } x_i = 0\,, \\ \underline{a}^2\underline{b}\underline{a}\underline{b} & \text{if } x_i = 1\,, \end{cases}$$

for each $i \in \mathbb{Z}^+$. It is not difficult to prove that ψ is a homeomorphism and that the diagram

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ \Sigma^* \arrow[r, "\sigma^r"] & \Sigma^* \end{tikzcd}$$

commutes. Therefore, by Corollary 2.2(c), Theorem 3.11 holds with X replaced by Σ^* and τ by σ^r.

Lemma 4.1. *Let $i, j \in \mathbb{Z}^+$. Then*

$$\sigma^i(\Sigma^*) \cap \sigma^j(\Sigma^*) = \emptyset$$

if and only if $i \not\equiv j$ modulo r.

Proof. Suppose that $\underline{\mathbf{d}} \in \sigma^i(\Sigma^*) \cap \sigma^j(\Sigma^*)$. Since $\sigma^r(\Sigma^*) = \Sigma^*$ we may assume that $0 \leq i < j < r$. Then, defining $k = j - i$, we obtain

$$\sigma^{r-i}(\underline{\mathbf{d}}) \in \sigma^r(\Sigma^*) \cap \sigma^{r+j-i}(\Sigma^*) = \Sigma^* \cap \sigma^k(\Sigma^*)$$

where $0 < k < r$. Hence $\sigma^{r-i}(\underline{\mathbf{d}}) \in \sigma^k(\Sigma^*)$ must begin either with $\underline{a}^3\underline{b}^2$ or with $\underline{a}^2\underline{b}\underline{a}\underline{b}$, which is impossible when $0 < k < r$ because $\underline{a}$ and $\underline{b}$ are different. The other implication is easy. ■

Now, since Theorem 3.11 holds with X replaced by Σ^* and τ by $\sigma^r|_{\Sigma^*}$, by Lemma 4.1 and Proposition 2.3 we obtain the following.

Propositon 4.2. *Theorem 3.11 holds with X and τ replaced by Σ and $\sigma|_\Sigma$, respectively.*

Before stating and proving Propositon 4.4 we need another technical lemma.

Lemma 4.3. *Assume that $g \in \mathcal{C}(I, I)$ is a P-monotone map such that $g|_P = f|_P$ and $\underline{\mathbf{d}} \in \mathcal{S}$ is g-followed by a point of P. Then $\underline{\mathbf{d}} \notin \Sigma$.*

Proof. Assume that there exist $x \in P$ and $\underline{\mathbf{d}} \in \Sigma$ such that $\underline{\mathbf{d}}$ is g-followed by x. Without loss of generality we may assume that $\underline{\mathbf{d}}$ begins with $\underline{a}^2\underline{b}$ (otherwise we replace x and $\underline{\mathbf{d}}$ by $g^n(x)$ and $\sigma^n(\underline{\mathbf{d}})$ for a suitable n). Then, clearly, x is an extremal point of $\kappa(l)$. Since $g^{|\underline{a}|}(x) \in \kappa(l)$, $g^{|\underline{a}|}(x)$ must be the other extremal point of $\kappa(l)$ because the period of P is strictly greater than $|\underline{a}|$. This implies that $g^{|\underline{a}|}(x) \neq x$ and thus, since $g^{2|\underline{a}|}(x)$ also belongs to $\kappa(l)$, we have $g^{2|\underline{a}|}(x) = x$.

Recall that we have $\underline{a} = (l, a_1, a_2, \dots)$ and $\underline{b} = (l, b_1, b_2, \dots)$ with $a_1 \neq b_1$. Then $g(x) \in \kappa(a_1)$ and $g(x) = g^{2|\underline{a}|+1}(x) \in \kappa(b_1)$. Hence, $g(x)$ is a common endpoint of two nonoverlapping intervals $\kappa(a_1)$ and $\kappa(b_1)$. On the other hand g is monotone on $\kappa(l)$, $x \in \kappa(l)$ is an endpoint of $\kappa(l)$ and $g(\kappa(l)) \supset \kappa(a_1) \cup \kappa(b_1)$ — a contradiction. ■

Proposition 4.4. *Let $f \in \mathcal{C}(I, I)$ be a map with positive topological entropy and let Σ and σ be defined as above. Then there exists a compact set $\mathcal{B} \subset I$ and a continuous map $\varphi : \mathcal{B} \longrightarrow \Sigma$ which is onto and one to one except for a countable set where it is two to one such that $f(\mathcal{B}) = \mathcal{B}$ and $\sigma \circ \varphi|_{\mathcal{B}} = \varphi \circ f|_{\mathcal{B}}$.*

Proof. For each $\underline{\mathbf{d}} = (d_0, d_1, \dots) \in \Sigma$ set $I_{\underline{\mathbf{d}}} = \cap_{i=0}^{\infty} f_P^{-i}(\kappa(d_i))$; that is, $I_{\underline{\mathbf{d}}}$ is defined to be the set of points which f_P-follow $\underline{\mathbf{d}}$. Then we set

$$\mathcal{B} = \bigcup_{\underline{\mathbf{d}} \in \Sigma} \mathrm{Bd}(I_{\underline{\mathbf{d}}}) .$$

We claim that

(a) Each $I_{\underline{\mathbf{d}}}$ is a (possibly degenerate) closed subinterval of $\langle P \rangle$.
(b) $\mathcal{B} \cap P = \emptyset$.
(c) $f_P(\mathcal{B}) = \mathcal{B}$.
(d) Each point in $\mathcal{B}$ f_P-follows a unique sequence from Σ.
(e) The sets $I_{\underline{\mathbf{d}}}$, for $\underline{\mathbf{d}} \in \Sigma$, are pairwise disjoint.

To prove the claim note that since f_P is P-monotone, (a) follows from the Itinerary Lemma. Statement (b) is a consequence of Lemma 4.3. Now observe that if $x \in I_{\underline{\mathbf{d}}}$ then $f_P(x) \in I_{\sigma(\underline{\mathbf{d}})}$. Hence, (c) follows from the fact that f_P is P-monotone. Therefore, (d) and (e) can be obtained immediately from (b). This ends the proof of the claim.

Now we will construct the map φ. For each $x \in \mathcal{B}$, we set $\varphi(x) = \underline{\mathbf{d}}$ if and only if $x \in I_{\underline{\mathbf{d}}}$. By (d), φ is well defined and, clearly, is continuous. Then, the fact that φ is onto and one to one except for a countable set where it is two to one can be proved by using (a) and (e).

The next step will be to prove that $\mathcal{B}$ is compact. First we will prove that the set

$$\mathcal{H} = \bigcup_{\underline{\mathbf{d}} \in \Sigma} I_{\underline{\mathbf{d}}}$$

is compact. To do it, for each $n \in \mathbb{Z}^+$, denote by Σ^n the set of finite sequences of the form $(d_0, d_1, \dots, d_n)$ for which there exists $\underline{\mathbf{d}} \in \Sigma$ which starts with $(d_0, d_1, \dots, d_n)$. Clearly, each of the sets Σ^n is finite. For each $\underline{a} = (a_0, a_1, \dots, a_n) \in \Sigma_n$ we denote by $I_{\underline{a}}^n$ the set $\cap_{i=0}^{n} f_P^{-i}(\kappa(a_i))$, which is a compact subset of I. Then, we have

$$\mathcal{H} = \bigcap_{n=0}^{\infty} \left(\bigcup_{\underline{a} \in \Sigma^n} I_{\underline{a}}^n \right) ,$$

which is clearly compact.

The continuity of f_P and (e) of the above claim imply that between two different sets $I_{\underline{\mathbf{d}}}$ there is a preimage of a point from P and, by Lemma 4.3, this point does not belong to $\mathcal{H}$. Therefore, it is not difficult to see that for each $x \in \mathcal{B}$ there exists a sequence contained in the complement of $\mathcal{H}$ which converges to x. Consequently, by the compacity of $\mathcal{H}$, $\mathcal{B} \subset \mathrm{Bd}(\mathcal{H})$. Thus, clearly, $\mathcal{B} = \mathrm{Bd}(\mathcal{H})$ and, hence, it is compact.

To prove that $f(\mathcal{B}) = \mathcal{B}$ assume that there exists $x \in \mathcal{B}$ such that $f_P(x) \neq f(x)$ then, by the definition of f_P, there is an open interval U containing x such that f_P is constant on U. Let $\underline{\mathbf{d}}$ be such that $x \in \mathrm{Bd}(I_{\underline{\mathbf{d}}})$. Clearly, $U \cup I_{\underline{\mathbf{d}}} \supsetneq I_{\underline{\mathbf{d}}}$ and all points in $U \cup I_{\underline{\mathbf{d}}}$ f_P-follow $\underline{\mathbf{d}}$ — a contradiction. Thus, from (c) it follows that $f(\mathcal{B}) = \mathcal{B}$.

To end the proof of the lemma notice that

$$\sigma \circ \varphi|_{\mathcal{B}} = \varphi \circ f_P|_{\mathcal{B}} = \varphi \circ f|_{\mathcal{B}} . \quad ■$$

Proof of Theorem A. At the end of Sec. 1 we have already seen that Conditions (a)–(c) imply $h(f) > 0$. So we assume that $h(f) > 0$ and we prove (a)–(c). Let $\{X_t\}_{t \in [0,1]}$ be the sets given by Theorem 3.11 with X and τ replaced by Σ and σ, respectively (see Proposition 4.2) and let φ be the map given by Proposition 4.4. Clearly, $\varphi^{-1}(X_t) \subset \mathcal{B}$ is compact for each $t \in [0, 1]$.

Now we state the following two elementary facts which are valid in any second countable topological space:

(i) Any closed set has a unique decomposition into a perfect set and a countable set (cf. Cantor–Bendixson Theorem).

(ii) The difference $A \setminus B$ of any perfect set A and any closed set B is either empty or uncountable (by a perfect set we understand here a closed set, locally uncountable at any point).

Thus, from (i) it follows that there is a countable set R_t such that $\omega_t = \varphi^{-1}(X_t) \setminus R_t$ is perfect and, since X_t is uncountable, nonempty. Since φ is continuous, $\varphi(\omega_t) \subset X_t$ is compact. On the other hand, $X_t \setminus \varphi(\omega_t) \subset \varphi(R_t)$ is a countable set. So, $\varphi(\omega_t) = X_t$ by (ii).

Now we claim that $f(\omega_t) \subset \omega_t$. Indeed, X_t is σ-invariant, hence, by Proposition 4.4 $\varphi^{-1}(X_t)$ is f-invariant. Therefore,

$$f(\omega_t) \setminus \omega_t \subset \varphi^{-1}(X_t) \setminus \omega_t \subset R_t$$

and so $f(\omega_t)\setminus\omega_t$ is countable. Note that σ is at most two-to-one on X_t. Hence, Proposition 4.4 implies that f is at most four-to-one on ω_t. Consequently, $f(\omega_t)$ is perfect and the claim follows from (ii).

By Proposition 4.2, there exists $z_t \in X_t$ such that $X_t = \omega_\sigma(z_t)$. So, there exists $x_t \in \omega_t \cap \varphi^{-1}(z_t)$. By the above claim, $\omega_f(x_t) \subset f(\omega_t) \subset \omega_t$. By Lemma 2.1 and Proposition 4.4,

$$\varphi(\omega_f(x_t)) = \omega_\sigma(\varphi(x_t)) = \omega_\sigma(z_t) = X_t\,.$$

So, since $\varphi(\omega_t \setminus \omega_f(x_t)) \subset X_t$, $\omega_t \setminus \omega_f(x_t)$ is countable by Proposition 4.4. Thus, $\omega_f(x_t) = \omega_t$ by (ii). Therefore, we have proved statement (a) and $\omega_t \subsetneq \omega_s$ for any $t < s$. Now, take $x \in I$ such that $\omega_f(x)$ is maximal with respect to the inclusion relation containing some ω_t. Since any two distinct maximal ω-limit sets of points for any continuous map of the interval have finite intersection (cf., e.g. [Schweizer *et al.*, 1994]), $\omega_s \subset \omega_f(x)$ for any $s \in I$. This ends the proof of (b). Finally, (c) follows from Proposition 4.2, Corollary 2.2 and Proposition 4.4. ■

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