

Minimum periods of homeomorphisms of orientable surfaces

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Certifiquem que la present memòria ha estat realitzada per la Moira Chas, i dirigida per nosaltres al Departament de Matemàtiques de la Universitat Autònoma de Barcelona.

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Dr. Lluís Alsedà i Soler Dr. Warren Dicks i McLay

To the beloved memory of Dante Leal,
who laughed, struggled and loved so much
in such a short time.

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Chapter 1

Introduction and main results

Given a set M and a map f from M to itself, recall the *orbit* of a point $x \in M$ is the set $\{x, f(x), f^2(x), \dots\}$. The data (M, f) is a *discrete dynamical system*, and the aim of the theory of dynamical systems is to understand the structure of the set of all orbits. The simplest kind of orbit is a *periodic orbit*, that is, the orbit of a point x for which there exists a positive integer n such that $f^n(x) = x$. If x is a point contained in a periodic orbit then it is called a *periodic point*, and the least n such that $f^n(x) = x$ is defined to be its *period*.

To obtain information about the structure of the set of orbits, we need to specify the data (M, f) . In general, M has some structure and one considers maps preserving this structure. This specification can be done in several ways. For instance, M can be a topological space and f a continuous map, or M a differentiable manifold and f a differentiable map, or M equipped with a σ -algebra structure and a measure and f preserves this measure or at least sets of measure zero.

Periodic orbits have always been an object of special interest in dynamical systems. Not only because of their simplicity but because their existence often has strong consequences for the dynamics of the map. In this sense, it is sometimes said that the set of orbits is the skeleton of the set of all orbits. Thus, another interesting object is the set of periods of f , which is the subset of \mathbb{N} consisting of the periods of all periodic points in M . This set is denoted by $\text{Per}(f)$.

It is interesting to deal with all the questions related to the periodic orbits in the field of *topological dynamics*, i.e., where M is a topological space and f a continuous map. Here, it is often possible to obtain useful information

about the structure of the set of orbits. In particular, much is known about the set $\text{Per}(f)$ when M has dimension one; see, for example, [2] and references therein. There are also many results for other spaces and classes of maps.

One of the main problems of the theory of dynamical systems is the determination of the existence of periodic orbits and, more generally the structure of $\text{Per}(f)$. We define the *minimum period of f* to be the maximum m , positive or infinite, such that the iterates f, f^2, \dots, f^{m-1} are fixed point free. We denote this number by $m(f)$. Observe that the minimum period of f is the greatest lower bound of $\text{Per}(f)$. If \mathcal{C} is a class of maps, the *minimum period of \mathcal{C}* is defined to be the maximum of the minimum periods of the mappings in \mathcal{C} , and is denoted by $m(\mathcal{C})$.

Throughout this thesis we are going to deal with some aspects of the above problem in the case of two-dimensional surfaces. More precisely, we will study the case where $M = \Sigma$, a connected orientable compact surface (possibly with boundary) and the maps of Σ we will consider are the orientation-preserving and orientation-reversing homeomorphisms. The reason for studying homeomorphisms of Σ and not just continuous maps is that for each surface Σ of genus g at least one there exists a map $f: \Sigma \rightarrow \Sigma$ with no periodic points, i.e., such that $m(f) = \infty$. Indeed, there exist a simple closed curve $\gamma \subset \Sigma$ and a map $g: \Sigma \rightarrow \gamma$ which is the identity on γ . There exists a map $h: \gamma \rightarrow \gamma$ conjugate to an irrational rotation $r: \mathbb{S}^1 \rightarrow \mathbb{S}^1$, so h has no periodic points. Clearly, we can view $h \circ g$ as a self-map of Σ and since $\text{Per}(h \circ g) = \emptyset$, $m(h \circ g) = \infty$.

Before going on, let us introduce some notation: The class of all (resp. all orientation-preserving, resp. all orientation-reversing) homeomorphisms of $\Sigma_{g,b}$ will be denoted by $\mathcal{H}_{g,b}$ (resp. $\mathcal{H}_{g,b}^+$, $\mathcal{H}_{g,b}^-$). Analogously, the class of all (resp. all orientation-preserving, resp. all orientation-reversing) homeomorphisms of Σ_g will be denoted by \mathcal{H}_g (resp. \mathcal{H}_g^+ , \mathcal{H}_g^-).

In the case of closed surfaces, the problem of determining the minimum period of the classes of orientation-preserving and orientation-reversing homeomorphisms is completely solved. The aim of this thesis is to study the minimum periods of homeomorphisms of surfaces with non-empty boundary, i.e., $m(\mathcal{H}_{g,b}^+)$ and $m(\mathcal{H}_{g,b}^-)$.

Let us return to the case of closed surfaces. Here, both bounds, $m(\mathcal{H}_g^+)$ and $m(\mathcal{H}_g^-)$ can be explicitly expressed as a function of the genus g . Their values are summarized in the following formulas.

$$(1.1) \quad m(\mathcal{H}_g^+) = \begin{cases} 1 & \text{if } g = 0, \\ \infty & \text{if } g = 1, \\ 2g - 2 & \text{if } g \geq 2. \end{cases}$$

$$(1.2) \quad m(\mathcal{H}_g^-) = \begin{cases} 2 & \text{if } g = 0, \\ \infty & \text{if } g = 1, \\ 4 & \text{if } g = 2, \\ 2g - 2 & \text{if } g \geq 2. \end{cases}$$

The first and well-known result in this field appeared in 1910. It is Brouwer's theorem [4] that an orientation-preserving homeomorphism of the sphere always has a fixed point. With the notation we have introduced, this result can be expressed as $m(\mathcal{H}_0^+) = 1$.

It is a simple matter to check that $m(\mathcal{H}_1^+)$ and $m(\mathcal{H}_1^-)$ are both infinite by exhibiting examples of orientation-preserving and orientation-reversing homeomorphisms of the torus Σ_1 with no periodic points. Indeed, view the torus Σ_1 as $\mathbb{S}^1 \times \mathbb{S}^1$, and consider the homeomorphism $(z, w) \rightarrow (ze^{\alpha i}, w)$, where α is an irrational real number. It is clear that this homeomorphism preserves and does not have fixed points. Similarly, the orientation reversing homeomorphism $(z, w) \rightarrow (ze^{\alpha i}, \bar{w})$ does not have periodic points.

In [35] Nielsen showed that $m(\mathcal{H}_g^+) = 2g - 2$ if $g \geq 3$. He also showed that $m(\mathcal{H}_2^+) \in \{2, 3\}$. The proof of these results uses the fixed-point theory due to Alexander [1] and Lefschetz [29] and [30] and some elementary algebra.

Later, Wang [40] showed that $m(\mathcal{H}_g^-) = 2g - 2$ if $g \geq 3$ and $m(\mathcal{H}_2^-) = 4$ by using methods analogous to these of Nielsen.

The problem about the determination of $m(\mathcal{H}_2^+)$, raised by Nielsen in 1942, remained open until 1996 when Dicks and Llibre [8] gave an algebraic proof that $m(\mathcal{H}_2^+) = 2$, which completes (1.1).

The only remaining case in (1.2) is $m(\mathcal{H}_0^-)$. Since the antipodal map is an orientation-reversing homeomorphism of Σ_0 which is fixed point free, $m(\mathcal{H}_0^-) \geq 2$. The equality $m(\mathcal{H}_0^-) = 2$ can be deduced from a theorem of Fuller [15]. This theorem states the existence of, and gives a bound for, the minimum periods of classes of homeomorphisms of compact ANRs. (See below for a definition of compact ANRs). In particular, it gives a general bound (and so the finiteness) for $m(\mathcal{H}_{g,b})$ except for two particular cases for which, as we shall see, the minimum period is ∞ . Before stating this result, we require some terminology.

A subset A of a topological metric space X is called an *compact absolute neighborhood retract* (or, briefly, *compact ANR*) if it has the following property: If A is a subspace of a separable metric space Y and A is homemorphic to X , then A is a neighborhood retract of Y .

If K is a compact ANR we denote by $H_k(K; \mathbb{Q})$ the k -th rational homology group of K . For each k , the dimension of $H_k(K, \mathbb{Q})$ is called the k -th *Betti number of K* and denoted by $b_k(K)$. The *Euler characteristic of K* is denoted by $\chi(K)$ and is defined to be $\sum (-1)^j b_j(K)$, a finite sum. Now we are ready to state Fuller's theorem. We will do it in the slightly more general version given in [5, Theorem III.E.2].

Theorem 1.1 *Let K be a compact ANR. If $\chi(K) \neq 0$ and $T: K \rightarrow K$ is a homeomorphism then*

$$m(T) \leq \max \left\{ \sum_{j \text{ odd}} (-1)^j b_j(K), \sum_{j \text{ even}} (-1)^j b_j(K) \right\}$$

In particular, surfaces are compact ANR's and their Betti numbers are

$$b_k(\Sigma_g) = \begin{cases} 1 & \text{if } k \in \{0, 2\}, \\ 2g & \text{if } k = 1, \\ 0 & \text{if } k \geq 3, \end{cases}$$

and

$$b_k(\Sigma_{g,b}) = \begin{cases} 1 & \text{if } k = 0, \\ 2g + b - 1 & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

Therefore, $\chi(\Sigma_g) = 2 - 2g$ and $\chi(\Sigma_{g,b}) = 2 - 2g - b$, so Fuller's theorem restricted to surfaces can be written in the following way:

Theorem 1.2 *(Fuller's Theorem for surfaces)*

- (1) *If $g \neq 1$ then $m(\mathcal{H}_g) \leq \max\{2, 2g\}$.*
- (2) *If $(g, b) \neq (0, 2)$ then $m(\mathcal{H}_{g,b}) \leq \max\{1, 2g + b - 1\}$.*

In particular, this result implies that $m(\mathcal{H}_{g,b}^+)$ and $m(\mathcal{H}_{g,b}^-)$ are finite whenever $(g, b) \neq (0, 2)$. Also, $m(\mathcal{H}_0^-) \leq 2$, which completes (1.2).

A subclass of homeomorphisms whose minimum period is also known is the class of finite-order maps. A homeomorphism $f: \Sigma \rightarrow \Sigma$ is said to be *finite-order* if there exists some positive integer n such that $f^n = \text{Id}_\Sigma$. The class of all (resp. all orientation-preserving, resp. all orientation-reversing) finite-order maps of Σ_g is denoted by \mathcal{F}_g (resp. \mathcal{F}_g^+ , \mathcal{F}_g^-). Nielsen [35] and Wang [40] determined the minimum period for \mathcal{F}_g^+ and \mathcal{F}_g^- , respectively. These results, together with the simple cases where $g \in \{0, 1\}$ are summarized in the following formulas.

$$m(\mathcal{F}_g^+) = \begin{cases} 1 & \text{if } g = 0, \\ \infty & \text{if } g = 1, \\ 2 & \text{if } g = 2, \\ g - 1 & \text{if } g \geq 3, \end{cases}$$

$$m(\mathcal{F}_g^-) = \begin{cases} 2 & \text{if } g = 0, \\ \infty & \text{if } g = 1, \\ 4 & \text{if } g = 2, \\ 2g - 2 & \text{if } g \geq 3. \end{cases}$$

1.1 Statement of the main results

Now we discuss the object of our study, surfaces with non-empty boundary.

It follows from Fuller's theorem 1.2(2) that $m(\mathcal{H}_{0,1}) = 1$ and $m(\mathcal{H}_{g,b}) \leq 2g + b - 1$ when $(g, b) \notin \{(0, 1), (0, 2)\}$. However, as we will see, except for $(g, b) \in \{(0, 1), (1, 1)\}$, the bounds given by that theorem are not the best possible. For example, the following gives a strictly smaller bound for $m(\mathcal{H}_{g,b})$ in most cases.

Proposition A *If $2g + b \geq 4$ then $m(\mathcal{H}_{g,b}) \leq 2g + b - 2$.*

We denote the class of all (resp. all orientation preserving, resp. all orientation reversing) finite-order maps of $\Sigma_{g,b}$ by $\mathcal{F}_{g,b}$ (resp. $\mathcal{F}_{g,b}^+$, $\mathcal{F}_{g,b}^-$). By exhibiting specific maps we will show that the bound given by Proposition A can be achieved if the pair (g, b) satisfies certain numerical conditions. Moreover, these maps are finite-order, so we have the following two theorems.

Theorem B *Let $g \geq 2$. Then $m(\mathcal{F}_{g,b}^+) = 2g + b - 2$ if and only if $b \in \{2, 3, 4\}$ or there exist positive integers p_1, p_2, p_3 such that they are pairwise coprime, each of them divides $2g + b - 2$ and $p_1 + p_2 + p_3 = b$.*

Theorem C *Let $g \geq 2$. Then $m(\mathcal{F}_{g,b}^+) = 2g + b - 2$ if and only if $b \in \{2, 4\}$ or one of the following conditions holds:*

- (1) *g is even and there exist positive integers p_1, p_2 such that each of them divides $2g + b - 2$, $\text{g.c.d}(p_1, p_2) = 2$ and $p_1 + p_2 = b$.*
- (2) *g is odd, b is even, and b divides $2g - 2$.*

By Proposition A, and Theorem B, $m(\mathcal{H}_{g,b}^+) = 2g + b - 2$ for certain pairs (g, b) . Also, if $g \geq 2$ it can be proved that there exists a homeomorphism $f \in \mathcal{H}_g^+ 1$ such that $m(f) = 2g - 1$ (and, clearly, $f \notin \mathcal{F}_g^+ 1$). Hence, we have the following.

Theorem D *Let $g \geq 2$. Then $m(\mathcal{H}_{g,b}^+) = 2g + b - 2$ if one of the conditions holds.*

- (1) *There exist positive integers p_1, p_2, p_3 such that they are pairwise coprime, each of them divides $2g + b - 2$ and $p_1 + p_2 + p_3 = b$.*
- (2) *$b - 2$ divides $2g$.*
- (3) *$b - 3$ divides $2g + 1$.*
- (4) *$b \in \{1, 2, 3, 4, g + 2, 2g + 2, 2g + 4\}$.*

Also, using Theorem C, we prove the following.

Theorem E *Let $g \geq 2$.*

- (1) *If b is odd then $m(\mathcal{H}_{g,b}^-) \leq b$ and equality holds if $b \leq 2g - 2$.*
- (2) *$m(\mathcal{H}_{g,b}^-) = 2g + b - 2$ if one of the following conditions holds.*
 - (i) *$b \in \{2, 4\}$.*
 - (ii) *g is odd, b is even and b divides $2g - 2$.*
 - (iii) *g is even, and there exist positive integers p_1, p_2 such that each of them divides $2g + b - 2$, $\text{g.c.d}(p_1, p_2) = 2$ and $p_1 + p_2 = b$.*
 - (iv) *g is even and $b - 2$ divides $2g$*
 - (v) *g is even and $b - 4$ divides $2g + 2$*
 - (vi) *g is even and $b \in \{g + 2, 2g + 2, 2g + 6\}$.*

b	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$m(\mathcal{H}_{2,b}^+)$	3	4	5	6	3	8	4	10	5	6	6	6	7	8
$m(\mathcal{H}_{2,b}^-)$	1	4	3	6	4	8	4	4	5	12	6	6	7	8

b	15	16	17	18	19	20	21	$b \geq 22$
$m(\mathcal{H}_{2,b}^+)$	8	8	9	10	10	10	10	10
$m(\mathcal{H}_{2,b}^-)$	8	8	8	8	9	10	11	12

Table 1.1: Values of $m(\mathcal{H}_{2,b}^+)$ and $m(\mathcal{H}_{2,b}^-)$.

We give values of the minimum periods for orientation-preserving (resp. orientation-reversing) homomorphisms of $\Sigma_{0,b}$, $\Sigma_{1,b}$, and $\Sigma_{2,b}$ in Theorem F (resp. Theorem G). Notice that in these cases, the bounds of Proposition A are achieved if and only if $g = 0$ and $b \geq 3$ (resp. $g \geq 2$), $g = 1$ and $b \geq 2$ (resp. $b \geq 1$), or $g \geq 2$ and b satisfies one of the conditions listed in Theorem D (resp. Theorem E).

Theorem F (1)

$$m(\mathcal{H}_{0,b}^+) = \begin{cases} 1 & \text{if } b = 1, \\ \infty & \text{if } b = 2, \\ b - 2 & \text{if } b \geq 3. \end{cases}$$

(2)

$$m(\mathcal{H}_{1,b}^+) = \begin{cases} 2 & \text{if } b = 1, \\ b & \text{if } b \geq 2. \end{cases}$$

(3) Table 1.1 shows the values of $m(\mathcal{H}_{2,b}^+)$.

Theorem G (1)

$$m(\mathcal{H}_{0,b}^-) = \begin{cases} 1 & \text{if } b = 1, \\ \infty & \text{if } b = 2, \\ 2 & \text{if } b = 3, \\ b - 2 & \text{if } b \geq 4. \end{cases}$$

(2) $m(\mathcal{H}_{1,b}^-) = b - 2$.

(3) Table 1.1 shows the values of $m(\mathcal{H}_{2,b}^-)$.

By Table 1.1, $m(\mathcal{H}_{2,b}^-) \leq 10$. We now explain roughly why this happens. (Precise arguments and definitions will be given later). Suppose that f is an orientation preserving homeomorphism of $\Sigma_{2,b}$. If some iterate of the map $\tilde{f}: \Sigma_2 \rightarrow \Sigma_2$ has a fixed point, of index different from one, then the same iterated of the original f has also a fixed point. On the other hand, it can be proved that for any $k: \Sigma_2 \rightarrow \Sigma_2$, there exists a positive integer n such that $n \leq 10$ and k^n has a fixed point of index different from one. So, $m(\mathcal{H}_{2,b}^-) \leq 10$. That equality can be achieved whenever $b \geq 18$ is shown by means of examples. The situation is analogous for any genus larger than or equal to 2, as is stated in Theorem H. To prove this, besides the ideas of [35], we also use the Thurston-Nielsen classification of surface homeomorphisms, and Nielsen fixed-point theory

Theorem H *If $g \geq 2$ then $m(\mathcal{H}_{g,b}^-) \leq 4g + 2$. Moreover, if $b \geq 6g + 6$, then equality holds.*

By Theorem H, the values of the minimum periods of orientation-preserving homeomorphisms of surfaces of genus at least 2 are bounded by a constant which does not depend on the number of boundary components. This situation is analogous for the classes of orientation reversing homeomorphisms as is stated in the following.

Theorem I *Let $g \geq 2$. Then $m(\mathcal{H}_{g,b}^-) \leq 4g + (-1)^g 4$ and equality holds if $b \geq 6g + 2 + (-1)^g 8$.*

Obviously, these theorems do not cover all possible cases. Indeed, if $g \geq 3$, the values of $m(\mathcal{H}_{g,b}^+)$ are not given if $b < 6g + 6$ and b does not satisfy the conditions of Theorem D. Similarly, the values of $m(\mathcal{H}_{g,b}^-)$ are not given if $g \geq 3$, $b < 4g + (-4)^g$, and b does not satisfy the conditions of Theorem E. the case $g = 3$ could be solved by completely analogous methods to teh used in Theorems F and G. However, when $g \geq 4$ the quantity of variables makes the calculations too complicated. Also, it is not clear that $m(\mathcal{H}_{g,b})$ expressed as a simple function of g and b .

This thesis is organized as follows. In Chapters 2 and 3 we give a summary of fixed-point theory and of the Thurston-Nielsen classification of homeomorphisms of surfaces, respectively. In Chapter 4 we present a standard form for such homeomorphisms. In Chapter 5 we present some features about planar discontinuous groups, and in Chapter 6 we apply these results about planar discontinuous groups to determine necessary and sufficient conditions for the

existence of certain finite-order maps of closed surfaces. Chapter 7 and 8 are devoted to developing the technical machinery which we will use in Chapters 9, 10 and 11 to prove our main results.

These thesis has three main branches, which are interconnected. One has to do with the application of fixed-point theory described in Chapter 2. All the upper bounds on the minimum periods except the ones stated in Theorems H and I are consequences of this theory. To obtain the upper bounds of Theorems H and I we apply also the Thurston-Nielsen classification of homeomorphisms and some of its consequences, described in Chapters 3 and 4. This is the second branch. Finally, then third branch has to do with the theory of planar discontinuous groups presented in Chapter 5, which will provide us with the tools for constructing examples which will show the existence of lower bounds for minimum periods.

Part I

Preliminary results

Chapter 2

Fixed-point theory

Let X be a space and let $f: X \rightarrow X$ be a self map. Fixed-point theory studies the nature of the set $\text{Fix}(f)$ in relation to the space X and the map f . This study can be undertaken from different points of view. Since we are dealing with homeomorphisms of topological manifolds, we focus our attention on the topological setting. Topological fixed-point theory tries to answer concerning $\text{Fix}(f)$, like what is the cardinal of this set, whether is it empty or not, or how does it change under homotopy.

Our aim is to study the existence of fixed points of iterates of continuous maps. Consequently, fixed-point theory provides very useful tools which are described in this chapter. More precisely, in Section 2.1 we give the definition of the Lefschetz number and state the Lefschetz Fixed-Point Theorem; Section 2.2 is devoted to index theory; and, in Section 2.3, we describe some particular features of the Lefschetz number for homeomorphisms of surfaces.

2.1 The Lefschetz Fixed-Point Theorem

Early in the history of fixed-point theory it was discovered that, if X is a polyhedron (see [5] for a definition), and $f: X \rightarrow X$ is a map with only a finite number of fixed points satisfying an additional technical requirement, it is possible to associate to each fixed point an integer, called the *index* which describes the way in which the map “winds around” the point. Furthermore, the sum of all indices was found to be equal to the *Lefschetz number*, which is defined for a continuous self-map f of a polyhedron X (or, more generally,

a compact ANR) as the finite sum

$$(2.1) \quad L(f) = \sum (-1)^k \operatorname{trace}(f_{*k}),$$

where $\operatorname{trace}(f_{*k})$ denotes the trace of the map f_{*k} induced by the action of f on the k -th rational homology group of X .

For every homeomorphism $f: X \rightarrow X$, every natural number k and every integer m , $(f_{*k})^m = (f^m)_{*k}$ (see [10, III.3]), and we shall write f_{*k}^m to denote their common value.

Remark 2.1 Observe that if $f, g: X \rightarrow X$ are two homotopic maps then for every natural number k , $f_{*k} = g_{*k}$. Therefore $L(f) = L(g)$. \square

Perhaps the best known fixed-point theorem in topology is the Lefschetz Fixed-Point Theorem.

Theorem 2.2 (Lefschetz (1923), Hopf (1929)) *Let X be a compact ANR and let $f: X \rightarrow X$ be a continuous map. If $L(f) \neq 0$ then every map homotopic to f has a fixed point.*

The first announcement of this theorem (for a restricted class of polyhedra) was in 1923 [29] and the details appeared in [30] and [31]. The first proof of the Lefschetz Fixed-Point Theorem for all polyhedra was given by Hopf [23]. Also, a particular case of this theorem valid for s -to-1 maps of surfaces follows from a theorem proved by Alexander [1] in 1923.

The Lefschetz number counts the fixed points “with multiplicity”. It is a homotopy invariant and is easily computable. As is the case in all the other literature about minimum periods, the Lefschetz Fixed-Point Theorem will be one of our main tools.

2.2 Index theory

The “multiplicity” of a fixed point (and, more generally, of a fixed-point set which is open in the whole set of fixed points of the map) is measured by the index. To study this index we shall develop some theory, following mainly [10, VII.5].

Our first step will be to define the index of certain types of map, firstly for a map from an open set of the Euclidean space \mathbb{R}^n to \mathbb{R}^n ; and subsequently

replacing \mathbb{R}^n with any ENR, (see the definition below). In order to achieve the former goal, we need to define a homological object: the fundamental class around a compact set. Before doing this, we need to recall some notions from algebraic topology.

Let X, Z be spaces, let Y (resp. W) be a subspace of X (resp. Z), and let $f: (X, Y) \rightarrow (Z, W)$ be a map. The k -th integer homology group of the pair (X, Y) is denoted by $H_k(X, Y; \mathbb{Z})$, and the map induced by f , from $H_k(X, Y; \mathbb{Z})$ to $H_k(Z, W; \mathbb{Z})$, is denoted by $H_k(f, \mathbb{Z})$. (Recall that the map induced by f on $H_k(X, \mathbb{Q})$ is denoted by f_{*k}).

Let n be a positive integer, let V be an open subset of \mathbb{R}^n , and let K be a compact subset of V . View \mathbb{S}^n as $\mathbb{R}^n \cup \{\infty\}$. Consider the inclusions

$$i: (\mathbb{S}^n, \emptyset) \rightarrow (\mathbb{S}^n, \mathbb{S}^n \setminus K)$$

and

$$j: (V, V \setminus K) \rightarrow (\mathbb{S}^n, \mathbb{S}^n \setminus K).$$

These maps induce homomorphisms

$$H_n(i, \mathbb{Z}): H_n(\mathbb{S}^n, \emptyset; \mathbb{Z}) \rightarrow H_n(\mathbb{S}^n, \mathbb{S}^n \setminus K; \mathbb{Z})$$

and

$$H_n(j, \mathbb{Z}): H_n(V, V \setminus K; \mathbb{Z}) \rightarrow H_n(\mathbb{S}^n, \mathbb{S}^n \setminus K; \mathbb{Z}).$$

By the Excision Lemma (see [10, Corollary III.7.4]), $H_n(j, \mathbb{Z})$ is an isomorphism. On the other hand, $H_n(\mathbb{S}^n, \emptyset; \mathbb{Z})$ is isomorphic to \mathbb{Z} , so we can fix one of its generators and denote it by o . We define the *fundamental class around K* , denoted by o_K , as the element $H_n(j, \mathbb{Z})^{-1} H_n(i, \mathbb{Z})(o)$ of $H_n(V, V \setminus K; \mathbb{Z})$.

Remark 2.1 o_K is characterized by the property that its image under the map induced by the inclusion

$$H_n(i, \mathbb{Z}): H_n(V, V \setminus K; \mathbb{Z}) \rightarrow H_n(V, V \setminus \{p\}; \mathbb{Z})$$

agrees with $o_{\{p\}}$ for every $p \in K$. Roughly speaking, it is an element of $H_n(V, V \setminus K; \mathbb{Z})$ which bounds K taking into account its orientation. \square

Let $V \subset \mathbb{R}^n$ be open and consider a map $f: V \rightarrow \mathbb{R}^n$ such that the set of fixed points, $\text{Fix}(f)$ is compact. Denote this set by K and let the map

$$i - f: (V, V \setminus K) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

be defined by $(i - f)(x) = x - f(x)$. Consider the homomorphism

$$H_n(i - f, \mathbb{Z}): H_n(V, V \setminus K; \mathbb{Z}) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}),$$

and define the *index of f* as the integer $I(f)$ such that

$$H_n(i - f, \mathbb{Z})(o_K) = I(f) \cdot o_{\{0\}}.$$

(Recall that $o_{\{0\}}$ generates $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$). This definition does not depend on the initial choice of the generator of $H_n(\mathbb{S}^n, \emptyset, \mathbb{Z})$ because $(-o)_{\{K\}} = -(o_{\{K\}})$ and $(-o)_{\{0\}} = -(o_{\{0\}})$.

Now we extend our definition of index to more general spaces, namely, Euclidean neighborhood retracts. A topological space Y is said to be a *Euclidean neighborhood retract* (or, briefly, an *ENR*) if a neighborhood retract $X \subset \mathbb{R}^n$ exists and is homeomorphic to Y . The following proposition will allow us to define the index for maps of ENR's (see [10, Proposition and Definition VII.5.10]).

Proposition 2.2 *If Y is a topological space and U is an open subset of Y which is also an ENR, then every map $f: U \longrightarrow Y$ admits a decomposition $f = \beta\alpha$ where $\alpha: U \longrightarrow V$, $\beta: V \longrightarrow Y$, and V is open in some Euclidean space \mathbb{R}^n .*

With the notation of the above proposition, consider the restrictions

$$\beta|_{\beta^{-1}(U)}: \beta^{-1}(U) \longrightarrow U \cap \beta(V)$$

and

$$\alpha|_{U \cap \beta(V)}: U \cap \beta(V) \longrightarrow V.$$

If $\text{Fix}(f)$ is compact we define the *index of f* , denoted by $I(f)$, as the index of

$$\alpha|_{U \cap \beta(V)} \circ \beta|_{\beta^{-1}(U)}: \beta^{-1}(U) \longrightarrow V.$$

The index defined in this way is independent of the decomposition (see [10, Proposition and Definition VII.5.10]).

Remark 2.3 Observe that for every ENR X , and every map $f: X \longrightarrow X$, if $\text{Fix}(f) = \emptyset$ then $H_k(V, V \setminus \text{Fix}(f); \mathbb{Z})$ is trivial for every natural number k . Hence, $I(f) = 0$. \square

If K is an open compact subset of $\text{Fix}(f)$ then we define the *index of K with respect to f* , denoted by $\text{Ind}_f(K)$, as the index of $f|_W$ where W is an open subset of U such that $K = \text{Fix}(f) \cap W$. This value is independent of the choice of W (see [10, VII.5.11]).

In particular, if $x \in X$ is an isolated fixed point then the set $\{x\}$ is compact and open in $\text{Fix}(f)$, so we can define its index, called the *index of f at x* and denoted by $\text{Ind}_f(x)$. There is an alternative way of calculating the index of a map at an isolated fixed point. Before describing it, we need to define another important notion of algebraic topology, the degree of a map.

Let M be an n -manifold and let $U \subset M$ be an open set. A map $f: U \rightarrow M$ is said to be *proper* if $f^{-1}(K)$ is compact for every compact set $K \subset M$. If the manifold M is connected and orientable, and $f: U \rightarrow M$ is a proper map, we define the *degree of f* , denoted $\deg(f)$, as the integer which satisfies

$$H_n(f, \mathbb{Z})(o_{f^{-1}(K)}) = \deg(f) \cdot o_K$$

for some non-empty compact subset K of M . It can be shown that $\deg(f)$ is independent of the choice of K (see [10, VIII.4]).

Now, we can state the promised alternative definition of index of a fixed point.

Remark 2.4 Let V be an open subset of \mathbb{R}^2 such that $\text{Cl}(\mathbb{D}^2) \subset V$. Assume that $f: V \rightarrow \mathbb{R}^2$ is a map such that the origin is the only fixed point of f in $\text{Cl}(\mathbb{D}^2)$. Define $\varphi_f: \partial\mathbb{D}^2 \rightarrow \partial\mathbb{D}^2$ by $\varphi_f(x) = \frac{x-f(x)}{|x-f(x)|}$. It is a simple exercise in algebraic topology to show that $\text{Ind}_f(x)$ equals the degree of φ_f . \square

There are other ways of defining the index of a fixed-point set although some of them require strong restrictions on the class of maps for which the definitions apply. In the general case, it is possible to give an axiomatic definition, (see [5]).

In the following remark, we give a property of the degree of a map of the circle.

Remark 2.5 If $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a map then $L(f) = 1 - \deg(f)$. In particular, if f is a homeomorphism,

$$L(f) = \begin{cases} 0 & \text{if } f \text{ preserves orientation,} \\ 2 & \text{if } f \text{ reverses orientation.} \end{cases}$$

\square

The following is a direct consequence of [10, VII.5.13].

Lemma 2.6 *If $C \subset \text{Fix}(f)$ is finite and open in $\text{Fix}(f)$ then $\text{Ind}_f(C)$ is equal to the sum of the indices of the elements of C .*

As mentioned above, the index of f on the whole of X equals $L(f)$, as is stated in the next theorem (see [10, VII.6.13]).

Theorem 2.7 *If X is a compact ENR and $f: X \rightarrow X$ is a map then*

$$L(f) = I(f).$$

Furthermore, if $U_i \subset X$, $i \in \{1, 2, \dots, k\}$ are open subsets such that $X = \bigcup_{i=1}^k U_i$ and for each pair $i \neq j$, $U_i \cap U_j \cap \text{Fix}(f) = \emptyset$ then

$$L(f) = \sum_{i=1}^k I(f|_{U_i}).$$

Consequently, if $\text{Fix}(f)$ is finite then $L(f)$ equals the sum of the indices of the fixed points.

By Remark 2.3, the Lefschetz Fixed-Point Theorem 2.2 is a consequence of Theorem 2.7. Observe that the definition of $I(f)$ uses integer homology groups whereas the definition of $L(f)$ uses rational homology groups.

For each homeomorphism $f: \Sigma \rightarrow \Sigma$, unless we specify the contrary, when we speak about the index of a fixed point of f^i (or a pointwise fixed set), we mean the index with respect to f^i .

In $\text{Fix}(f)$ we define the following relation: Given $x, y \in \text{Fix}(f)$, we say that x and y are *f-equivalent* if there is a path α from x to y such that α and $f \circ \alpha$ are homotopic keeping the endpoints fixed. It is easy to see that the relation of *f-equivalence* is an equivalence relation. The equivalence classes are called *fixed-point classes of f*. It can be proved (see [5], [25]) that a fixed-point class is compact and open in $\text{Fix}(f)$, so its index is defined. A fixed-point class is called *essential* if its index is different from 0. Essential fixed-point classes will be important for us because they survive (preserving their index) under isotopy, as is stated in the following theorem, which is a corollary of [5, Theorem VI.E.3].

Theorem 2.8 *Let i be an integer different from 0, let X be a compact polyhedron, and let $f: X \rightarrow X$ be a map. If f has a fixed-point class of index i and $g: X \rightarrow X$ is homotopic to f , then g has a fixed-point class of index i .*

Another important property of fixed-point classes is given in the following theorem (see, for instance, [25, Theorem I.4.3 and Theorem I.4.4]).

Theorem 2.9 *The number of essential fixed-point classes is finite and the sum of the indices of all (essential) fixed-point classes of f equals $L(f)$.*

We end this section by stating the following proposition (see [10, Exercise VII.6.25.2]).

Proposition 2.10 *Let X, A be compact ENR's such that $A \subset X$, and let $f: (X, A) \rightarrow (X, A)$ be a map. Then*

$$L(f) + 1 = L(\bar{f}) + L(f|_A),$$

where $\bar{f}: X/A \rightarrow X/A$ denotes the map induced by f on the quotient space X/A .

2.3 Lefschetz numbers of maps of surfaces

If $f: \Sigma \rightarrow \Sigma$ is a homeomorphism, the Lefschetz number of f , $L(f)$, takes a particular form. Here $\text{trace}(f_{*0}) = 1$ and

$$\text{trace}(f_{*2}) = \begin{cases} 1 & \text{if } f \text{ preserves orientation and } \partial\Sigma = \emptyset, \\ -1 & \text{if } f \text{ reverses orientation and } \partial\Sigma = \emptyset, \\ 0 & \text{if } \partial\Sigma \neq \emptyset, \end{cases}$$

so (2.1) can be rewritten as

$$(2.2) \quad L(f) = \begin{cases} 2 - \text{trace}(f_{*1}) & \text{if } f \text{ preserves orientation and } \partial\Sigma = \emptyset, \\ -\text{trace}(f_{*1}) & \text{if } f \text{ reverses orientation and } \partial\Sigma = \emptyset, \\ 1 - \text{trace}(f_{*1}) & \text{if } \partial\Sigma \neq \emptyset. \end{cases}$$

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of f_{*1} . For each positive integer i , $L(f^i) = \text{trace}(f_{*0}^i) + \text{trace}(f_{*2}^i) - \sum_{j=1}^n \lambda_j^i$. Therefore, the polynomials

$$p_i = p_i(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{j=1}^n \lambda_j^i$$

play a key role in the computation of the numbers $L(f^i)$. Indeed, by (2.2),

$$(2.3) \quad L(f^i) = \begin{cases} 2 - p_i & \text{if } f^i \text{ is orientation-preserving and } \partial\Sigma = \emptyset, \\ -p_i & \text{if } f^i \text{ is orientation reversing and } \partial\Sigma = \emptyset, \\ 1 - p_i & \text{if } \partial\Sigma \neq \emptyset. \end{cases}$$

Clearly, the sequence of p_i 's determines the sequence of $L(f^i)$'s. For this reason, we will study in more detail some of its properties. Namely, we will study Newton's equations.

Let k be a positive integer and let $M \in GL_k(\mathbb{Z})$. Denote by $\lambda_1, \lambda_2, \dots, \lambda_k$ the eigenvalues of M . If we write the characteristic polynomial of M as

$$P(x) = \det(xI_k - M) = x^k + s_1x^{k-1} + s_2x^{k-2} + \dots + s_{k-1}x + s_k,$$

then, for each $j \in \{1, 2, \dots, k\}$,

$$s_j = (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq k} \lambda_{i_1} \cdots \lambda_{i_j}, \text{ and}$$

$$(N.1) \quad p_1 + s_1 = 0,$$

$$(N.2) \quad p_2 + s_1p_1 + 2s_2 = 0,$$

$$\vdots$$

$$(N.k) \quad p_k + s_1p_{k-1} + s_2p_{k-2} + \dots + ks_k = 0, \text{ and}$$

$$(N.(k+1)) \quad p_{k+l} + s_1p_{k+l-1} + s_2p_{k+l-2} + \dots + s_kp_l = 0 \text{ for } l \geq 1.$$

See, for example, [38, Exercise 2, Section 29].

If $f: \Sigma \rightarrow \Sigma$ is a homeomorphism then the characteristic polynomial of f_{*1} will be called the *characteristic polynomial of f* .

Now, in order to study one of the properties of f_{*1} , we shall introduce more notation. For each positive integer g denote by J_g the element

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

of $GL_{2g}(\mathbb{Z})$. We say that $M \in GL_{2g}(\mathbb{Z})$ is *proper symplectic* (resp. *improper symplectic*) if $M^t J_g M = J_g$ (resp. $M^t J_g M = -J_g$). A matrix M is *symplectic* if it is either proper symplectic or improper symplectic.

A property of proper symplectic matrices is stated in the following; see [36].

Proposition 2.1 *If $M \in GL_{2g}(\mathbb{Z})$ is proper symplectic then $\det(M) = 1$.*

The next result is a consequence of [43, Theorem 3.6.7].

Proposition 2.2 *With respect to a certain basis, for every $f \in \mathcal{H}_g^+$ (resp. \mathcal{H}_g^-), the matrix of f_{*1} is proper symplectic (resp. improper symplectic).*

Chapter 3

The Thurston-Nielsen classification

The goal of this chapter is to give a brief introduction to the Thurston-Nielsen classification of isotopy classes of surface homeomorphisms. This is, undoubtedly, the most important tool in the topological theory of surface dynamics. It can be viewed as a prime decomposition theorem: it gives the existence in each isotopy class of a homeomorphism that is constructed by gluing together homeomorphisms of two types, pseudo-Anosov and finite-order. The theory has numerous applications and implications for many diverse areas of mathematics, but we will focus on some of its dynamical aspects. The reader is referred to [37] for the original proof, to [13] or [20] for proofs of the theorem for orientable surfaces, and to [42] for a proof for non-orientable surfaces. Also, an algorithmic proof can be found in [3].

This chapter is organized in the following way: Sections 3.1 and 3.2 are devoted to finite-order maps and to pseudo-Anosov maps respectively, and, in Section 3.3 we define reducible maps, give the Thurston-Nielsen classification and state some properties of finite-order maps.

3.1 Finite-order maps

The simplest types of maps used in the construction of a Thurston representative are the finite-order maps defined in Chapter 1. They are dynamically very simple: the period of each orbit equals the order n of the map, except for a finite number of orbits whose period is a divisor of n . If $f: \Sigma \rightarrow \Sigma$

is an isometry with respect to a hyperbolic metric, then it is standard that f is finite-order; see [13, Exposé 3, Théoreme 18]. Conversely, when f is finite-order on a surface of negative Euler characteristic, it is an isometry with respect to some hyperbolic metric; see [12, Theorem 2.8].

If $f: \Sigma \rightarrow \Sigma$ is a homeomorphism, then the *order of f* , denoted by σ_f , is the least positive integer σ such that $f^\sigma = \text{Id}_\Sigma$, or ∞ if no such σ exists. The *order* of a non-empty class \mathcal{C} of homeomorphisms is defined to be the supremum of the σ_f , $f \in \mathcal{C}$. Wiman [41] determined the order of \mathcal{F}_g^+ and Wang [39] determined the order of \mathcal{F}_g^- ; see also [21]. These results are summarized in the following.

Theorem 3.1 *If $g \geq 2$ then the order of \mathcal{F}_g^+ is $4g + 2$ and the order of \mathcal{F}_g^- is $4g + (-1)^g 4$.*

Remark 3.2 Observe that the order of \mathcal{F}_g^+ (resp. \mathcal{F}_g^-) coincides with the upper bound given for $m(\mathcal{H}_{g,b}^+)$ (resp. $m(\mathcal{H}_{g,b}^-)$) by Theorem H (resp. Theorem I). \square

As well as the order of finite-order maps, we require some information about the connected components of $\text{Fix}(f)$. This is stated in the following lemma which is a consequence of [27, Lemma 1.1]; see also [12, Theorem 2.8].

Lemma 3.3 *If $f: \Sigma \rightarrow \Sigma$ is an orientation-preserving finite-order homeomorphism, and A is a connected component of $\text{Fix}(f)$, then either $A = \Sigma$, or A contains only one point and it has a neighborhood homeomorphic to \mathbb{D}^2 where f acts as a rotation. In the former case, $\text{Ind}_f(A) = \chi(\Sigma)$, and in the latter case, $\text{Ind}_f(A) = 1$.*

Observe that the trace of $(\text{Id}_\Sigma)_{*k}$ equals the k -th Betti number of Σ , i.e., $\text{trace}((\text{Id}_\Sigma)_{*k}) = b_k(\Sigma)$ for each $k \in \mathbb{N}$. In particular, by (2.2), $L(\text{Id}_\Sigma) = \chi(\Sigma)$. By Lemma 3.3 and Theorem 2.7, we have the following.

Lemma 3.4 *If $f: \Sigma \rightarrow \Sigma$ is an orientation-preserving finite-order map then*

$$(3.1) \quad L(f) = \begin{cases} \text{Card}(\text{Fix}(f)) & \text{if } f \neq \text{Id}_\Sigma, \\ \chi(\Sigma) & \text{if } f = \text{Id}_\Sigma. \end{cases}$$

Now we state a result analogous to Lemma 3.3 for the orientation-reversing case; see [12, Theorem 2.8] for the proof.

Lemma 3.5 *Let $f \in \mathcal{F}_g^-$ and A be a connected component of $\text{Fix}(f)$. Then A is a simple closed curve, with a neighborhood U homeomorphic to $\mathbb{S}^1 \times (0, 1)$ where f acts as the reflection $(z, t) \mapsto (z, 1 - t)$. Moreover, $\text{Ind}_f(A) = 0$.*

Applying Lemmas 3.3 and 3.5 we obtain the following.

Lemma 3.6 *Let $f \in \mathcal{F}_g^-$. If there exists $i \in \{1, 2, \dots, \sigma_f - 1\}$ such that $f^i|_C = \text{Id}_C$ for some simple closed curve $C \subset \Sigma_g$, then $\sigma_f \equiv_4 2$ and $i = \sigma_f/2$.*

3.2 Pseudo-Anosov maps

A detailed description of pseudo-Anosov maps of surfaces without boundary can be found in [16], but this author could not find a good description for pseudo-Anosov maps of surfaces with boundary. For this reason we give a complete definition here, although for our purposes it would be enough to give the description of a standard form defined in Chapter 4.

A *singular foliation* F of Σ is a partition of $\Sigma \setminus \{x_1, x_2, \dots, x_m\}$, for some finite subset $\{x_1, x_2, \dots, x_m\}$ of Σ , into a disjoint union of one-dimensional manifolds, called *leaves*, such that there exists a finite C^∞ atlas with charts

$$(\varphi_i, U_i)_{1 \leq i \leq l},$$

where $\varphi_i: U_i \rightarrow \mathbb{C}$, with $m \leq l$ and $\cup_{1 \leq i \leq l} U_i = \Sigma$, and a finite sequence of integers $p_i \geq 3$, $i \in \{1, 2, \dots, m\}$, and $s \in \{0, 1, 2, \dots, m\}$, such that the following statements hold:

Case 1. If $1 \leq i \leq s$ then

- (1) $\varphi_i(U_i) = D_{a_i} \cap \{z \in \mathbb{C} : \Re(z) \geq 0\}$ for some $a_i > 0$;
- (2) $x_i \in U_i$ and $\varphi_i(x_i) = 0$;
- (3) For each leaf L of F , if K is a component of $L \cap U_i$, then there exists $k \in [0, \infty)$ such that K is mapped bijectively to a component of $\{z \in \mathbb{C} : \Im(z^{p_i-1}) = k\} \cap \varphi_i(U_i)$.

Case 2. If $s + 1 \leq i \leq m$ then

- (1) $\varphi_i(U_i) = D_{a_i}$ for some $a_i > 0$;
- (2) $x_i \in U_i$ and $\varphi_i(x_i) = 0$;

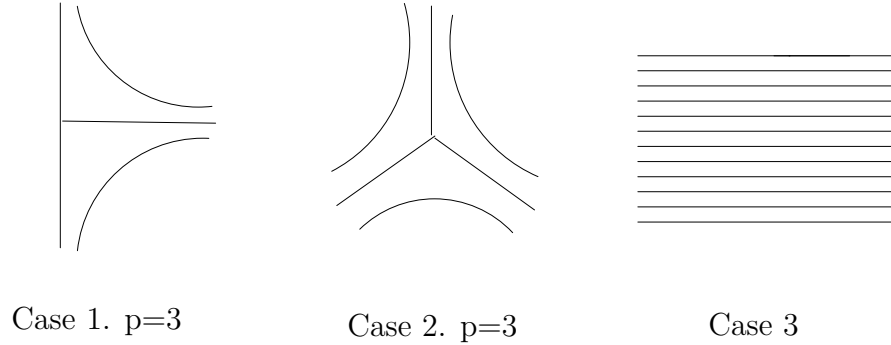


Figure 3.1: Charts for singular foliations

- (3) For each leaf L of F , if K is a component of $L \cap U_i$, then there exists $k \in [0, \infty)$ such that K is mapped bijectively to a component of $\{z \in \mathbb{C} : \Im(z^{p_i/2}) = k\} \cap \varphi_i(U_i)$.

Case 3. If $m + 1 \leq i \leq l$ then

- (1) $\varphi_i(U_i) = (0, b_i) \times (0, c_i)$ or $(0, b_i) \times [0, c_i)$ for some $b_i, c_i > 0$;
- (2) For each leaf L of F , if K is a component of $L \cap U_i$, then there exists $k \in [0, \infty)$ such that K is mapped bijectively to a component of $\{z \in \mathbb{C} : \Im(z) = k\} \cap \varphi_i(U_i)$.

The points $x_i, i \in \{1, 2, \dots, m\}$, are called the *singularities* of the foliation and the points in $\Sigma \setminus \{x_1, x_2, \dots, x_m\}$ are called *regular points*. Observe that each boundary component of Σ is a finite union of leaves and singular points.

For each $i \in \{1, 2, \dots, m\}$ there are p_i leaves emanating from x_i . These leaves are called *prongs* and we refer to x_i as a *p-pronged singularity*. If x is a regular point then it is contained in a single leaf L . However for convenience, we will refer to the oriented components of $L \setminus \{x\}$ as the *prongs emanating from x* and to x as a *2-pronged point*. A leaf emanating from a boundary

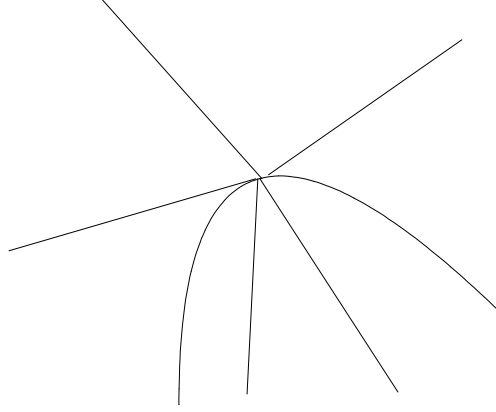


Figure 3.2: An arc transverse to a foliation in a singularity

component B but not contained in B is called a *prong of B* and B is called a *p -pronged boundary component* if there are exactly p prongs emanating from it.

Let x be a singularity of a singular foliation F and let (φ_x, U_x) be a chart such that $x \in U_x$. We say that an arc $\alpha \subset \Sigma$ such that $x \in \alpha$ is *transverse to F in x* if there exists $r > 0$ such that for every $\bar{r} < r$, $(\alpha \setminus \{x\}) \cap \varphi_x^{-1}(D_{\bar{r}})$ has exactly two connected components and each of them is included in one connected component of $\varphi_x^{-1}(D_{\bar{r}} \cap (\varphi_x(U_x) \setminus \{z \in \mathbb{C} : \Im(z^{p/2}) = 0\}))$; see Figure 3.2.

An arc α is *transverse to a foliation F* if it is transverse to the leaves of F in the usual sense in the regular points of F and transverse to F in the singularities.

A *transverse measure μ to a foliation with singularities F* is a map which assigns to each arc α transverse to F a non-negative Borel measure μ_α on α , with the following properties:

- (1) If β is a subarc of α then the measure μ_β is the restriction of μ_α to β .
- (2) If α_0, α_1 are two arcs in Σ transverse to F related by a homotopy $\alpha: [0, 1] \times [0, 1] \rightarrow \Sigma$ such that $\alpha([0, 1] \times \{0\}) = \alpha_0$ and $\alpha([0, 1] \times \{1\}) = \alpha_1$ and $\alpha([0, 1] \times \{a\})$ is contained in a leaf of F for each $a \in [0, 1]$, then $\mu_{\alpha_0} = \mu_{\alpha_1}$.

If $k: \Sigma \rightarrow \Sigma$ is a homeomorphism and F is a foliation of Σ we say that

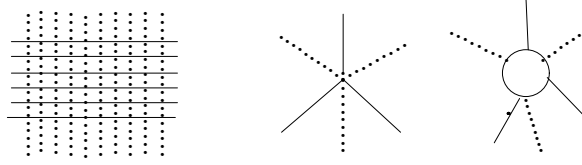


Figure 3.3: Charts for transverse singular foliations

F is *invariant under k* if the images under k of the leaves of F are leaves of F . If F is a foliation invariant under k and μ is a measure transverse to F , we define the *image measure* $k(\mu)$ as the transverse measure to F such that if α and β are arcs transverse to F then $k(\mu)_\beta(\gamma) = \mu_{k^{-1}(\beta)}(k^{-1}(\gamma))$ for every Borel set $\gamma \subset \alpha$.

A pair (F, μ) is a *measured foliation* if F is a singular foliation and μ is a transverse measure to F .

Let F_1 and F_2 be two singular foliations and assume that x is an interior p -pronged singularity for F_1 and F_2 . We say that F_1 and F_2 are *transverse in x* if there exists a neighborhood U of x , $r > 0$, and a homeomorphism $\varphi: U \rightarrow D_r$ such that

- (1) $\varphi(x) = 0$;
- (2) For each leaf L of F_1 , if K is a component of $L \cap U$, then there exists $k \in [0, \infty)$ such that K is mapped bijectively to a component of $\{z \in \mathbb{C} : \Re(z^{p/2}) = k\} \cap \varphi(U)$; see Figure 3.3
- (3) For each leaf L of F_2 , if K is a component of $L \cap U$, then there exists $k \in [0, \infty)$ such that K is mapped bijectively to a component of $\{z \in \mathbb{C} : \Im(z^{p/2}) = k\} \cap \varphi(U)$; see Figure 3.3

Let B be a boundary component of Σ , and let F_1 and F_2 be two singular foliations on Σ . Collapsing B to a point x , the foliations F_1 and F_2 induce two (not necessarily singular) foliations \overline{F}_1 and \overline{F}_2 respectively in the resultant surface $\overline{\Sigma}$. We say that F_1 and F_2 are *transverse in B* if \overline{F}_1 and \overline{F}_2 are transverse in x ; (see Figure 3.3). Two singular foliations are *transverse* if they have the same interior singularities, and they are transverse in interior singularities, boundary components and regular points (see Figure 3.3).

A map $f: \Sigma \rightarrow \Sigma$ is called *pseudo-Anosov* if there exist two measured transverse foliations (F^s, μ^s) and (F^u, μ^u) which are invariant under f , and

such that each boundary component contains at least a singular point, and a real constant $\lambda > 1$ such that $f(\mu^s) = \lambda^{-1}\mu^s$ and $f(\mu^u) = \lambda\mu^u$. (This is usually denoted as $f(F^s, \mu^s) = (F^s, \lambda^{-1}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$). This number λ is called the *expansion constant* for f .

Remark 3.1 If $\alpha \subset \Sigma$ is an arc included in a leaf of F^u (resp. F^s) and it is not included in the boundary of Σ then α is transverse to F^s (resp. F^u). Thus, $\mu^s(\alpha)$ (resp. $\mu^u(\alpha)$) is defined. Furthermore, $\mu^s(f(\alpha)) = \lambda^{-1}\mu^s(\alpha)$ (resp. $\mu^u(f(\alpha)) = \lambda\mu^u(\alpha)$).

On the other hand, if α is an arc included in $\partial\Sigma$ then neither $\mu^s(\alpha)$ nor $\mu^u(\alpha)$ are defined. \square

From now on we will work with only the unstable foliation F^u . For simplicity, we will refer to it as *the* foliation.

Denote by $\text{Sing}(\Sigma)$ the set of singularities of the foliation. A very useful equation which relates the Euler characteristic of Σ with the foliation is the *Euler-Poincaré Formula*,

$$(3.2) \quad \sum_{s \in \text{Sing}(\Sigma)} (2 - p_s) = 2\chi(\Sigma),$$

where p_s is the number of prongs emanating from s for each $s \in \text{Sing}(\Sigma)$; see [13, exposé 5] for a proof.

Remark 3.2 If B is a p -pronged boundary component, then

$$\sum_{s \in \text{Sing}(B)} (2 - p_s) = -p.$$

\square

3.3 The classification theorem

A *system of reducing curves* for a surface Σ is a finite (possibly empty) set of pairwise disjoint simple closed curves $\Gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_n\} \subset \Sigma$ such that each connected component of $\Sigma \setminus \Gamma$ has negative Euler characteristic. If $f: \Sigma \rightarrow \Sigma$ is a homeomorphism, then a *system of reducing curves* for f , or an *f -system of reducing curves*, is a system of reducing curves Γ for Σ

which is f -invariant and has an f -invariant tubular neighborhood $N(\Gamma)$ of Γ , called an f -*tubular neighborhood*. A homeomorphism $f: \Sigma \rightarrow \Sigma$ is said to be *reducible* if there exists a non-empty system of reducible curves Γ for f and, for each connected component R of $\Sigma \setminus N(\Gamma)$, there exists a positive integer n such that:

- (1) $f^n(R) = R$.

- (2) $f^n|_R: R \rightarrow R$ is either finite-order or pseudo-Anosov.

The subsurface R is called an f -*component* or *component of f* . The least positive integer n which satisfies $f^n(R) = R$ is called the *period of R* or f -*period of R* . We say that a R is a *pseudo-Anosov component of f* or a *finite-order component of f* if the homeomorphism $f^n|_R: R \rightarrow R$ is pseudo-Anosov or finite-order, respectively.

Theorem 3.1 (Thurston-Nielsen) *If $f: \Sigma \rightarrow \Sigma$ is a homeomorphism then f is isotopic to a homeomorphism f' which is finite-order, pseudo-Anosov or reducible.*

Here we say that f' is in *Thurston canonical form*. Notice that f' is not uniquely determined by f . In general, one says that an isotopy class is *finite-order*, *pseudo-Anosov* or *reducible* if an element in Thurston canonical form has the corresponding property. If an isotopy class is finite-order then any complicated behavior of the maps in this class can be isotoped away. However, complicated behavior of maps in a pseudo-Anosov class persists under isotopy.

Chapter 4

The standard form

Any isotopy class contains infinitely many homeomorphisms as Thurston canonical forms. However, while finite-order isotopy classes have, roughly speaking, a unique representative, for pseudo-Anosov homeomorphisms of surfaces with boundary, there is a certain amount of choice involved in the structure of the foliation and the dynamics of the representatives of a given class. There are a number of papers that have used this freedom to refine the Thurston canonical form for dynamical purposes. Such a refinement was used to prove the existence of a dynamically minimal representative for Nielsen classes of fixed points in the category of surface homeomorphisms. This result was sketched by Jiang in [26] and given in full detail by Jiang and Guo in [27]. They isotope the Thurston canonical form in different steps. We are not interested in the final refinement but in a specific type of Thurston canonical form obtained during the process. They call this particular type *the standard form*.

In order to prove our main results we need to prove the existence of fixed-points classes of index different from one for iterates of a homeomorphism. Since essential fixed-point classes (and their indices) are preserved under isotopy, it will be enough to prove that each map in an isotopy class has a fixed-point class of index 0, or that one representative of the isotopy class has a fixed-point class of negative index. We will do this for maps in standard form because this class of maps has very useful properties: iterates of a map in standard form are also in standard form, fixed-point classes are connected, their structure is well understood, and their indices are easily calculable.

Throughout this chapter we will follow [27], in order to give a description of the standard form and to state some of its properties. This description

will be done in the spirit of the Thurston-Nielsen Theorem: In Section 4.1 we describe the standard form for pseudo-Anosov maps and its fixed-point classes; in Section 4.2 we define a reducible map in standard form to be a reducible map in Thurston canonical form. The main result of this chapter is Proposition 4.4, where we describe the structure of the fixed-point classes of a map in standard form.

4.1 Standard form for pseudo-Anosov maps

In view of the need to glue finite-order pieces and pseudo-Anosov pieces together, the standard form is required to be finite-order on $\partial\Sigma$. By definition, a map in standard form is smooth but we are not going to take this property into account here, because it will not be necessary for our purposes.

We begin by defining some auxiliary maps. Fix a real number $\lambda > 1$ and a natural number p . Consider the correspondence $\Phi_p: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$, $z \mapsto z^{p/2}$, and the diffeomorphism $\Psi: \mathbb{C} \setminus \mathbb{D}^2 \rightarrow \mathbb{C} \setminus \{0\}$ defined as

$$\Psi(z) = z - \frac{z}{|z|}.$$

Let \overline{V} be the vector field on \mathbb{C} obtained by “slowing down” the vector field

$$V(s) = \lambda \overline{s},$$

with a non-decreasing smooth function $\alpha: \mathbb{R} \rightarrow [0, 1]$ such that $\alpha(0) = 0$ and $\alpha(1/\lambda^2) = 1$; that is, $\overline{V}(s) = \alpha(|s|)V(s)$.

Recall that the time-one map associated to a vector field W is defined as $\varphi(x, 1)$, where $\varphi(x, t)$ is the solution of the equation

$$\begin{cases} \frac{d\varphi(x, t)}{dt} = W(\varphi(x, t)) \\ \varphi(x, 0) = x, \end{cases}$$

and recall that the vector field $\Phi_*(\overline{V})$ is defined in a point q as the product of the matrix $d\Phi_{\phi(q)}^{-1}$ with the vector $\overline{V}(q)$.

Let $f: \mathbb{C} \rightarrow \mathbb{C}$ and $f': \mathbb{C} \setminus \mathbb{D}^2 \rightarrow \mathbb{C} \setminus \mathbb{D}^2$ be the time-one maps associated to the vector fields

$$v_p(z) = \begin{cases} \Phi_{p*}^{-1} \overline{V}(z) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

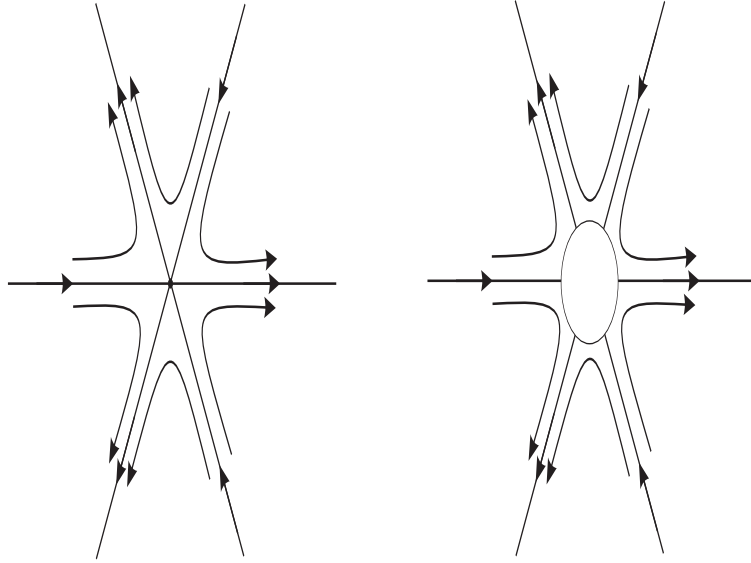


Figure 4.1: The flows of v_p and v'_p ($p = 3$).

$$v'_p(z) = \begin{cases} \Psi_*^{-1}v_p(z) & \text{if } z \in \mathbb{C} \setminus \text{Cl}(\mathbb{D}^2), \\ 0 & \text{if } z \in \partial\mathbb{D}^2. \end{cases}$$

See Figure 4.1.

Now, consider $\alpha \in \mathbb{R}$ and define $r_\alpha^+ : \mathbb{C} \rightarrow \mathbb{C}$ and $r_\alpha^- : \mathbb{C} \rightarrow \mathbb{C}$ by

$$r_\alpha^+(z) = ze^{2i\alpha\pi} \text{ and } r_\alpha^-(z) = \bar{z}e^{-2i\alpha\pi}.$$

We say that a pseudo-Anosov map $f : \Sigma \rightarrow \Sigma$ is in *standard form* if there is a finite smooth atlas U of Σ , consisting of one chart for each interior singularity, one chart for each boundary component and charts at interior regular points such that:

- (1) If x is an interior p -pronged point (possibly regular) and (U_x, u_x) is the chart for x then the measures μ^s and μ^u are mapped by u_x to the

measures $|\Re d\Phi_p(z)|$ and $|\Im d\Phi_p(z)|$ on \mathbb{C} respectively. The leaves of F^s and F^u are mapped to the lines $\{z \in \mathbb{C} : \Re \Phi_p(z) = \text{constant}\}$ and $\{z \in \mathbb{C} : \Im \Phi_p(z) = \text{constant}\}$ respectively.

- (2) If A is a p -pronged boundary component and (U_A, u_A) is its chart, then the measures μ^s and μ^u on U_A get mapped to the measures $|\Re d\Phi_p\Psi(z)|$ and $|\Im d\Phi_p\Psi(z)|$ on $\mathbb{C} \setminus \mathbb{D}^2$ respectively. The leaves of F_s and F_u get mapped to the lines $\{z \in \mathbb{C} : \Re \Phi_p\Psi(z) = \text{constant}\}$ and $\{\Im \Phi_p\Psi(z) = \text{constant}\}$ respectively.
- (3) For each chart (U_x, u_x) at an interior singularity x , $u_x(U_x)$ contains the closed disk $\text{Cl}(\mathbb{D}^2)$. Moreover $\text{Cl}(\mathbb{D}^2) \cap U_y = \emptyset$ for every $y \neq x$ such that (U_y, u_y) is a chart of the atlas. Similarly for the closed annulus $\{z : 1 \leq |z| \leq 2\}$ in each chart (U_A, u_A) .
- (4) If x is a p -pronged point or A is a p -pronged boundary component then for each chart (U_x, u_x) and (U_A, u_A) , there exists $0 \leq k < p$ such that the following diagrams commute.

$$\begin{array}{ccc}
(U_x, \{x\}) & \xrightarrow{f} & (U_{\varphi(x)}, \{\varphi(x)\}) & (U_A, A) & \xrightarrow{f} & (U_{\varphi(A)}, \varphi(A)) \\
\downarrow u_x & & \downarrow u_{\varphi(x)} & \downarrow u_A & & \downarrow u_{\varphi(A)} \\
(\mathbb{C}, \{0\}) & \xrightarrow{r_{k/p}^\epsilon f} & (\mathbb{C}, \{0\}) & (\mathbb{C} \setminus \mathbb{D}^2, \{0\}) & \xrightarrow{r_{k/p}^\epsilon f'} & (\mathbb{C} \setminus \mathbb{D}^2, \{0\})
\end{array}$$

where ϵ is $+$ or $-$ when the map f is orientation-preserving or reversing, respectively.

We say that a fixed point x or an invariant boundary component A is of *type* $(p, k)^+$ (resp. of *type* $(p, k)^-$) if f preserves orientation (resp. reverses orientation) and one of the following statements holds:

- (1) x is a singularity or A is a boundary component and $(p, k)^+$ (resp. $(p, k)^-$) is as in part (4) of the definition of standard form.
- (2) x is a regular point, $(p, k) = (2, 0)$ and the two prongs in F^u emanating from x remain fixed under the action of f .
- (3) x is a regular point, $(p, k) = (2, 1)$ and the two prongs in F^u emanating from x are interchanged by f .

Roughly speaking, we can say that a pseudo-Anosov map acts as a rotation of angle $2k\pi/p$ on the prongs of a fixed point of type $(p, k)^+$ and as a reflection on the prongs of a fixed point of type $(p, k)^-$. So we have the following remark.

Type of x	$(p, 0)^+$	$(p, k)^+$	$(2n, 0)^-$	$(2n, 1)^-$	$(2n + 1, 0)^-$
$\text{Ind}_f(x)$	$1 - p$	1	-1	1	0

Table 4.1: Indices of fixed points of a pseudo-Anosov map

Type of A	$(p, 0)^+$	$(p, k)^+$	$(2n, 0)^-$	$(2n, 1)^-$	$(2n + 1, 0)^-$
$\text{Fix}(f) \cap A$	a	\emptyset	2 points	2 points	2 points
$\text{Ind}_f(A)$	$-p$	0	$0 + 0$	$1 + 1$	$1 + 0$

Table 4.2: Indices of invariant boundary components of a pseudo-Anosov map

Proposition 4.1 *Let x (resp. A) be a fixed point (resp. an invariant boundary component) of type $(p, k)^+$ of an orientation-preserving pseudo-Anosov homeomorphism f . Then, for each positive integer n , x (resp. A), is of type $(p, \widetilde{nk}^p)^+$ for f^n . In particular, x (resp. A) is of type $(p, 0)^+$ for f^p . In other words, all prongs emanating from x (resp. A) remain fixed under the action of f^p .*

On the other hand, if x (resp. A) is a fixed point (resp. an invariant boundary component) of an orientation-reversing pseudo-Anosov map f of type $(p, k)^-$ then x (resp. A) is of type $(p, 0)^+$ for f^2 . Hence, all prongs emanating from x (resp. A) remain fixed under the action of f^2 .

Proposition 4.2 *Let $f: \Sigma \longrightarrow \Sigma$ be a pseudo-Anosov homeomorphism in standard form.*

- (1) *If f preserves orientation then the fixed-point classes of f are either interior fixed points or invariant boundary components.*
- (2) *If f reverses orientation then the fixed-point classes of f are fixed points.*

Furthermore, the indices of fixed-point classes depend on their types as described in Tables 4.1 and 4.2.

4.2 Definition and properties of the standard form

We say that a homeomorphism $f: \Sigma \longrightarrow \Sigma$ in Thurston canonical form is *in standard form* if one of the following holds

- (i) f is finite-order
- (ii) f is a pseudo-Anosov map in standard form.
- (iii) f is reducible, every component of f satisfies (i) or (ii), and for each connected component A of $N(\Gamma)$, if n is a positive integer such that $f^n(A) = A$, then $f^n|_{\text{Cl}(A)}$ is conjugate to one of the following maps of $\mathbb{S}^1 \times I$.
 - (1) $(z, t) \mapsto (ze^{2(a+bt)\pi i}, t)$, where a and b are rational numbers.
 - (2) $(z, t) \mapsto (\bar{z}e^{a(1-2t)\pi i}, 1-t)$, where a is rational.
 - (3) $(z, t) \mapsto (ze^{2a\pi i}, 1-t)$ where a is a rational number.
 - (4) $(z, t) \mapsto (\bar{z}, t)$; where \bar{z} denotes the conjugate of z .

For each connected component A of $N(\Gamma)$, the minimum positive integer m such that $f^m(A) = A$ is called the *f-period* of A .

Now we state properties of the standard form.

Lemma 4.1 *Each isotopy class of homeomorphisms of a surface contains a map in standard form.*

The following can easily be deduced from the definition of standard form.

Lemma 4.2 *Iterates of a map in standard form are also in standard form.*

Remark 4.3 If $f: \Sigma \longrightarrow \Sigma$ is an orientation-preserving map in standard form and B is an f -invariant boundary component of Σ then $f|_B$ is a rotation. Furthermore, if B belongs to a pseudo-Anosov component of f and p is the number of prongs of B then $f^p|_B = \text{Id}_B$ and all the prongs emanating from B remain fixed under the action of f^p . \square

A description of the fixed-point classes of a map in standard form is given in the next result, which plays a key role in the proofs of Theorems H and I.

Proposition 4.4 ([27, Lemma 3.6]) *Let $f: \Sigma \longrightarrow \Sigma$ be a map in standard form. Assume that C is a fixed-point class of f . Then one of the following holds.*

- (A) $C = \{x\}$ where x is an isolated fixed point, f preserves orientation and one of the following holds:
- (A.1) $x \in \text{Int}(\Sigma)$ and f is conjugate to a rotation in a neighborhood of x and $\text{Ind}_f(x) = 1$.
 - (A.2) $x \in \text{Int}(\Sigma)$ is a fixed point of a connected component A of $N(\Gamma)$, $f|_{\text{Cl}(A)}$ is conjugate to a map of the form $(z, t) \mapsto (\bar{z}e^{a(1-2t)\pi i}, 1-t)$ and $\text{Ind}_f(x) = 1$.
 - (A.3) $x \in \text{Int}(\Sigma)$ is a type $(p, k)^+$ interior fixed point of a pseudo-Anosov component and $\text{Ind}_f(x) = 1 - p$ if $k = 0$ and $\text{Ind}_f(x) = 1$ otherwise.
- (B) $C = \{x\}$ where x is an isolated fixed point and f reverses orientation and one of the following holds:
- (B.1) $x \in \text{Int}(\Sigma)$ is an interior fixed point of a pseudo-Anosov component and $\text{Ind}_f(x) \in \{1, -1, 0\}$.
 - (B.2) $x \in \partial\Sigma$ is in a type $(p, k)^-$ invariant boundary component of some pseudo-Anosov component and $\text{Ind}_f(x) = 1$ or 0 .
- (C) C is a simple closed curve, f is orientation-preserving and one of the following holds:
- (C.1) $C \subset \text{Int}(\Sigma)$ and $C \subset A$ for some connected component A of $N(\Gamma)$, $f|_{\text{Cl}(A)}$ is conjugate to a map of the form $(z, t) \mapsto (ze^{2(a+bt)\pi i}, t)$ and $\text{Ind}_f(C) = 0$
 - (C.2) $C \subset \text{Int}(\Sigma)$, C is a type $(p, 0)^+$ boundary component of a pseudo-Anosov component of f , and $\text{Ind}_f(C) = -p$.
 - (C.3) $C \subset \partial\Sigma$, C is a type $(p, 0)^+$ boundary component of a pseudo-Anosov component of f and $\text{Ind}_f(C) = -p$.
- (D) C is a simple closed curve and f is orientation-reversing, $C \subset \text{Int}(\Sigma)$, in a neighborhood of C f is conjugate to the reflection $(z, t) \mapsto (z, 1-t)$ and $\text{Ind}_f(C) = 0$.
- (E) f is orientation-reversing, and C is a fixed arc. C is contained in a subsurface B on which f acts as an involution (i.e., $f^2 = \text{Id}$). Moreover, if x is an endpoint of C such that $x \in \text{Int}(\Sigma)$ then it is in a boundary component of a pseudo-Anosov component. Also, $\text{Ind}_f(x) \in \{1, -1, 0\}$.
- (F) f is orientation-preserving and C is a fixed subsurface of Σ with $\chi(C) \leq 0$. If B is a boundary component of C such that $B \subset \text{Int}(\Sigma)$ then B is also a boundary component of either a component of $N(\Gamma)$ or a

component of a pseudo-Anosov component of f . In the latter case, B is of type $(p, 0)^+$. Moreover $\text{Ind}_f(C) = \chi(C) - \sum p_B < 0$ where the summation is over the components B of ∂C which are also boundary components of a pseudo-Anosov component of f and p_B is the number of prongs emanating from B .

Chapter 5

Planar discontinuous groups

The goal of this chapter is to give a brief exposition of some features of the theory of planar discontinuous groups. This theory will provide us the necessary tools for the construction and study of finite-order homeomorphisms of closed surfaces in Chapter 6.

Let us begin by introducing some notation and definitions. We shall denote by ϑ an element of the set $\{+, -\}$. A *signature* consists of a sign $+$ or $-$ and an ordered sequence of integers with certain subsequences bracketed together in the following manner:

$$(\vartheta, T, [m_1, m_2, \dots, m_R], \{(m_{1,1}, m_{1,2}, \dots, m_{1,M_1}), \\ (m_{2,1}, m_{2,2}, \dots, m_{2,M_2}), \dots (m_{B,1}, m_{B,2}, \dots, m_{B,M_B})\})$$

and satisfying:

- (1) $T \geq 0$, $R \geq 0$, $B \geq 0$, and $M_i \geq 0$ for each $i \in \{1, 2, \dots, B\}$. Moreover, if the sign of the signature is $+$, T is even;
- (2) If $T = B = 0$ then the sign of the signature is $+$;
- (3) For every $i \in \{1, 2, \dots, R\}$, $m_i \geq 2$ and $m_i \leq m_j$ if $i \leq j$;
- (4) For every $i \in \{1, 2, \dots, B\}$, $j \in \{1, 2, \dots, M_i\}$, $m_{i,j} \geq 2$.

Observe that R , M_i for some $i \in \{1, 2, \dots, B\}$ or B can be 0. In such cases, the signature will be written with the brackets inserted, but with no symbols between them. In fact, the signatures which we will consider in Lemma 5.11, and afterwards, satisfy $M_i = 0$ for every $i \in \{1, 2, \dots, B\}$. To abbreviate the notation we shall denote these signatures by

$$(\vartheta, T, [m_1, m_2, \dots, m_R], B).$$

If

$$\Psi = (\vartheta, T, [m_1, m_2, \dots, m_R], \{(m_{1,1}, m_{1,2}, \dots, m_{1,M_1}), \\ (m_{2,1}, m_{2,2}, \dots, m_{2,M_2}), \dots (m_{B,1}, m_{B,2}, \dots, m_{B,M_B})\})$$

is a signature, the *Euler characteristic* of Ψ is defined as

$$\mu(\Psi) = 2 - T - B - \sum_{i=1}^R \left(1 - \frac{1}{m_i}\right) - \frac{1}{2} \sum_{i=1}^B \sum_{j=1}^{M_i} \left(1 - \frac{1}{m_{i,j}}\right).$$

We say that a group G is a *(cocompact) planar discontinuous group* if there exists a signature Ψ as in the above paragraph such that $\mu(\Psi) \leq 0$ and G has a presentation with generators:

- (1) $\sigma_i, i \in \{1, 2, \dots, R\},$
- (2) $\tau_i, i \in \{1, 2, \dots, T\},$
- (3) $\pi_i, i \in \{1, 2, \dots, B\},$
- (4) $\rho_{i,j}, i \in \{1, 2, \dots, B\}, j \in \{1, \dots, M_i + 1\};$

and defining relations

- (1) $\sigma_i^{m_i}, i \in \{1, 2, \dots, R\},$
- (2) $\rho_{i,j}^2, i \in \{1, 2, \dots, B\}, j \in \{1, 2, 3, \dots, M_i + 1\},$
- (3) $(\rho_{i,j} \rho_{i,j+1})^{m_{i,j}}, i \in \{1, 2, \dots, B\}, j \in \{1, 2, \dots, M_i\},$
- (4) $\pi_i^{-1} \rho_{i,M_i+1} \pi_i \rho_{i,1}, i \in \{1, 2, \dots, B\};$
- (5) (i) If $\vartheta = +$ then

$$\pi_1 \pi_2, \dots, \pi_B \sigma_1 \sigma_2 \dots, \sigma_R [\tau_1, \tau_2] [\tau_3, \tau_4] \dots [\tau_{T-1}, \tau_T]$$

(where $[a, b]$ denotes $aba^{-1}b^{-1}$).

- (ii) If $\vartheta = -$ then

$$\pi_1 \pi_2, \dots, \pi_B \sigma_1 \sigma_2 \dots, \sigma_R \tau_1^2 \tau_2^2 \dots, \tau_T^2.$$

It can be proved that a planar discontinuous group G determines its signature up to certain permutations, see [34, Theorems 1a, 2a and 3]. We require a weaker version of this result:

Theorem 5.5 *If the signatures*

$$(\vartheta, T, [m_1, \dots, m_R], \{(m_{1,1}, \dots, m_{1,M_1}), \dots (m_{B,1}, \dots, m_{B,M_B})\}) \text{ and}$$

$$(\tilde{\vartheta}, \tilde{T}, [\tilde{m}_1, \dots, \tilde{m}_R], \{(\tilde{m}_{1,1}, \dots, \tilde{m}_{1,M_1}), \dots, (\tilde{m}_{B,1}, \dots, \tilde{m}_{B,M_B})\})$$

are associated to a planar discontinuous group G then $\vartheta = \tilde{\vartheta}$, $T = \tilde{T}$, $R = \tilde{R}$, $m_i = \tilde{m}_i$ for each $i \in \{1, 2, \dots, R\}$, $M_i = \tilde{M}_i$ for each $i \in \{1, 2, \dots, B\}$ and

$$\{m_{i,1}, m_{i,2}, \dots, m_{i,M_i}\} = \{\tilde{m}_{i,1}, \tilde{m}_{i,2}, \dots, \tilde{m}_{i,M_i}\}$$

for each $i \in \{1, 2, \dots, B\}$.

In particular, the order of each of the generators is determined by the relations (1) and (2), and, for each $i \in \{1, 2, \dots, B\}$, $j \in \{1, 2, \dots, M_i\}$, the order of $(\rho_{i,j}\rho_{i,j+1})$ is $m_{i,j}$.

Remark 5.6 The requirement that $m_i \leq m_j$ if $i \leq j$ in the definition of a signature could have been dropped. In this case, the m_i are uniquely determined up to permutation. \square

Let G be a planar discontinuous group with signature Ψ . We define the *Euler characteristic* of G as

$$\mu(G) = \mu(\Psi).$$

By Theorem 5.5, the Euler characteristic of a planar discontinuous group is well defined. A planar discontinuous group is said to be *non-Euclidean* (resp. *Euclidean*) if $\mu(G) < 0$ (resp. $\mu(G) = 0$).

Given a planar discontinuous group G , we define a homomorphism

$$\varepsilon: G \longrightarrow \{-1, 1\},$$

called the *orientation map*, which acts on the generators in the following way:

$$\begin{aligned} \varepsilon(\sigma_i) &= 1 && \text{for each } i \in \{1, 2, \dots, R\}, \\ \varepsilon(\pi_i) &= 1 && \text{for each } i \in \{1, 2, \dots, B\}, \\ \varepsilon(\rho_{ij}) &= -1 && \text{for each } i \in \{1, 2, \dots, B\}, j \in \{1, 2, \dots, M_i\}, \\ \varepsilon(\tau_i) &= \begin{cases} 1 & \text{if } \vartheta = +, \\ -1 & \text{otherwise.} \end{cases} \end{aligned}$$

A planar discontinuous group G is said to be *orientable* if $\varepsilon(G) = \{1\}$ and *non-orientable* if $\varepsilon(G) = \{-1, 1\}$. By Theorem 5.5 this definition is independent of the signature of G .

A *surface group* is a planar discontinuous group for which $B = R = 0$.

Remark 5.7 Notice that if G is a surface group, then it is isomorphic to the fundamental group of a surface. More precisely, if G is orientable (resp. non-orientable) and has signature $(\vartheta, T, [], \{ \})$, then it is isomorphic to $\pi_1(\Sigma_{\frac{T}{2}})$ (resp. $\pi_1(N_T)$). Here, $\pi_1()$ denotes the fundamental group, and N_T a non-orientable compact connected closed surface of genus T . Observe that if G is orientable then T is even and

$$\mu(G) = \chi(\Sigma_{\frac{T}{2}}),$$

and if G is non-orientable then

$$\mu(G) = \chi(N_T).$$

□

Now we state the well-known Riemann-Hurwitz Formula (see, for instance, [43, Theorem 4.14.22])

Theorem 5.8 *Let G be a planar discontinuous group and let H be a subgroup of G of finite index. Then H is a planar discontinuous group and*

$$\mu(H) = [G : H]\mu(G).$$

Moreover, the orientation of H is the restriction of the orientation of G .

Here, $[G : H]$ denotes the index of H in G .

Remark 5.9 Observe that, by Theorem 5.8, if H is a subgroup of finite index of a non-Euclidean (resp. Euclidean) planar discontinuous group G then H is a non-Euclidean (resp. Euclidean) planar discontinuous group. Further, if G is orientable so is H . □

As usual, if G and H are groups and $\phi: G \longrightarrow H$ is a homomorphism, $\text{Ker}(\phi)$ denotes the kernel of ϕ . If H is finite, it follows from Theorem 5.8 that $\text{Ker}(\phi)$ is also a planar discontinuous group. In the following lemma we determine necessary and sufficient conditions for $\text{Ker}(\phi)$ be a surface group. The proof is elementary and can be found in [21] for the particular case of non-Euclidean planar discontinuous orientable groups. The general case is analogous and for this reason, we do not include it in this exposition.

Lemma 5.10 *Let $\phi: G \longrightarrow H$ be a homomorphism where G is a planar discontinuous group and H is a finite group. Then ϕ preserves the orders of the elements of finite order in G if and only if $\text{Ker}(\phi)$ is a surface group.*

In the following lemma, we state a property which must be satisfied for planar discontinuous groups G which admit epimorphisms $\phi: G \longrightarrow \mathbb{Z}_n$ preserving the orders of its finite-order elements.

Lemma 5.11 *Let G be a planar discontinuous group with signature Ψ and let n be a positive integer. If there exists an epimorphism $\phi: G \longrightarrow \mathbb{Z}_n$ which preserves the orders of finite-order elements of G then $M_i = 0$ for each $i \in \{1, 2, \dots, B\}$.*

Proof. Assume that $M_i \geq 1$ for some $i \in \{1, 2, \dots, B\}$. Since $\rho_{i,1}^2 = \rho_{i,2}^2 = 1$, n must be even and $\phi(\rho_{i,1}) = \phi(\rho_{i,2}) = [\frac{n}{2}]_n$. Hence, $\rho_{i,1}\rho_{i,2} \in \text{Ker}(\phi)$. On the other hand, by Theorem 5.5, the order of $\rho_{i,1}\rho_{i,2}$ is $m_{i,1} \geq 2$, a contradiction. ■

From now on, we are going to consider only planar discontinuous groups G which admit epimorphisms $\phi: G \longrightarrow \mathbb{Z}_n$ for some positive integer n preserving the orders of finite-order elements of G . By Lemma 5.11, the signature of such groups can be denoted by $(\vartheta, T, [m_1, m_2, \dots, m_R], B)$.

Let n, B, R and $1 \leq p_1, p_2, \dots, p_R < n$ be elements of \mathbb{N} . We say that a finite-order homeomorphism $f: \Sigma \longrightarrow \Sigma$ is of *type* $[n; B; \{p_1, p_2, \dots, p_R\}]$ if the following holds.

- (1) $n = \sigma_f$;
- (2) For each $i \in \{1, 2, \dots, R\}$ there exists a periodic f -orbit O_i of period p_i . Furthermore, $O_i \cap O_j = \emptyset$ if $i \neq j$;
- (3) There exist B simple closed curves C_1, C_2, \dots, C_B in Σ such that
 - (i) For each $1 \leq i, j \leq n/2$, and for each $1 \leq l, m \leq B$,

$$f^i(C_l) \cap f^j(C_m) = \emptyset, \text{ if } l \neq m \text{ or } i \neq j.$$

- (ii) $f^{\frac{n}{2}}|_{C_i} = \text{Id}_{C_i}$ for each $i \in \{1, 2, \dots, B\}$;

- (4) Every point $x \in \Sigma \setminus \left(\bigcup_{i=1}^B \bigcup_{j=1}^{\frac{n}{2}} f^j(C_i) \cup \bigcup_{i=1}^R O_i \right)$ has period n .

The following result will be crucial for our arguments, because, combined by Harvey's Theorem 5.14 and Theorem 6.2, will give necessary and sufficient conditions for the existence of a finite-order map of a given type. The proof can be found in [18] for the particular case of orientable planar discontinuous groups and orientable homeomorphisms of prime order. This proof can be easily generalized for orientable groups and homeomorphisms of any order. The more general statement we give here is a consequence of some results of [43, Chapter 4].

Theorem 5.12 *Let n be a positive integer, let G be a planar discontinuous group with signature*

$$(\vartheta, T, [m_1, m_2, \dots, m_R], B)$$

and let $\phi: G \longrightarrow \mathbb{Z}_n$ be an epimorphism such that $\text{Ker}(\phi)$ is an orientable surface group. Then there exists a finite-order homeomorphism $f: \Sigma \longrightarrow \Sigma$ of type

$$[n; B; \{\frac{n}{m_1}, \frac{n}{m_2}, \dots, \frac{n}{m_R}\}].$$

where Σ is a closed surface such that $\pi_1(\Sigma)$ is isomorphic to $\text{Ker}(\phi)$. Moreover, the genus of Σ equals

$$1 + \frac{n(T + B + R - 2) - \sum_{i=1}^R n/m_i}{2}.$$

Conversely, let g be a positive integer and let $f: \Sigma_g \longrightarrow \Sigma_g$ be a finite-order homeomorphism of type $[n; B; \{p_1, p_2, \dots, p_R\}]$. Then

$$T = 2 - B - R + \frac{2g - 2 + \sum_{i=1}^R p_i}{n}$$

is a non-negative integer and f determines an epimorphism from a planar discontinuous group G of signature $(\vartheta, T, [\frac{n}{p_1}, \frac{n}{p_2}, \dots, \frac{n}{p_R}], B)$ to \mathbb{Z}_n such that $\text{Ker}(\phi)$ is isomorphic to $\pi_1(\Sigma_g)$.

In both cases, f is orientation-preserving if and only if G is orientable.

By Theorem 5.12 (or by Lemma 3.3) we have the following.

Corollary 5.13 *If there exists $f \in \mathcal{F}_g^+$ of type*

$$[n; B; \{p_1, p_2, \dots, p_R\}]$$

then $B = 0$.

The next theorem, due to Harvey [21], determines when, given a signature Ψ and an integer $n \geq 2$, there exists an epimorphism from an orientable group G with signature Ψ to \mathbb{Z}_n such that its kernel is a surface group.

Theorem 5.14 (Harvey) *Let $n \geq 2$. Suppose that G is an orientable planar discontinuous group of signature*

$$(+, T, [m_1, m_2, \dots, m_R], B)$$

and let

$$M = \text{l.c.m.}(m_1, m_2, \dots, m_R).$$

Then there exists an epimorphism $\phi: G \longrightarrow \mathbb{Z}_n$ such that $\text{Ker}(\phi)$ is a surface group if and only if the following conditions are satisfied:

- (1) $\text{l.c.m.}(m_1, m_2, \dots, \hat{m}_i, \dots, m_R) = M$ for each $i \in \{1, 2, \dots, R\}$ where \hat{m}_i denotes the omission of m_i ;
- (2) M divides n , and, if $T = 0$, $M = n$;
- (3) $R \neq 1$ and, if $T = 0$, $R \geq 3$;
- (4) if 2^m divides M , and 2^{m+1} does not divide M for some positive integer m , the cardinal of the set $\{i \in \{1, 2, \dots, R\} : 2^m \text{ divides } m_i\}$ is even.

The following proposition is the analogue of Theorem 5.14 for orientable Euclidean groups.

Lemma 5.15 *Let G be a Euclidean planar discontinuous orientable group. Then there exists an epimorphism $\phi: G \longrightarrow \mathbb{Z}_n$ such that $\text{Ker}(\phi)$ is a surface group if and only if one of the following holds,*

- (1) The signature of G is $(+, 0, [2, 2, 2, 2], 0)$ and $n = 2$.
- (2) The signature of G is $(+, 0, [3, 3, 3], 0)$ and $n = 3$.
- (3) The signature of G is $(+, 0, [2, 4, 4], 0)$ and $n = 4$.
- (4) The signature of G is $(+, 0, [2, 3, 6], 0)$ and $n = 6$.
- (5) The signature of G is $(+, 2, [], 0)$.

Proof. A simple calculation shows that if $\mu(G) = 0$ then the signature of G is one of those listed in (1)-(5); see [34, Table I]. Thus, the result follows from Lemma 5.10. ■

Part II

Development of the tools

Chapter 6

Finite-order maps of closed surfaces

This chapter is dedicated to the study of homeomorphisms of finite order of closed surfaces. This study has two goals: Firstly, to give a characterization of the possible types of maps in \mathcal{F}_g^+ and in \mathcal{F}_g^- which will allow us to determine, in Chapter 9, $m(\mathcal{F}_{g,b}^+)$ and $m(\mathcal{F}_{g,b}^-)$. The second goal is the construction of maps with “large” minimum periods in Chapters 9 and 10.

The organization of this chapter is as follows: in Sections 6.1 and 6.2 we determine necessary and sufficient conditions for the existence of a map with a given type in \mathcal{F}_g^+ and \mathcal{F}_g^- , respectively. In Section 6.3 we construct some special maps in \mathcal{F}_g^+ and \mathcal{F}_g^- .

6.1 The orientation-preserving case

The following result is a consequence of Theorem 5.12 and Harvey’s Theorem 5.14.

Corollary 6.1 *Let $g \geq 2$, $n \geq 2$, and $1 \leq p_1, p_2, \dots, p_R < n$, be positive integers. Set*

$$d = \text{g.c.d.}(p_1, p_2, \dots, p_R) \text{ and } T = \frac{2g - 2 + \sum_{i=1}^R p_i}{n} - R + 2.$$

Then there exists $f \in \mathcal{F}_g^+$ of type $[n; 0; \{p_1, p_2, \dots, p_R\}]$ if and only if the following conditions hold.

- (1) T is a non-negative even integer;
- (2) $\text{g.c.d}(p_1, p_2, \dots, \hat{p}_i, \dots, p_R) = d$ for each $i \in \{1, 2, \dots, R\}$;
- (3) For each $i \in \{1, 2, \dots, R\}$, p_i divides n ;
- (4) If $T = 0$, $d = 1$;
- (5) $R \neq 1$ and, if $T = 0$, $R \geq 3$.

A necessary and sufficient condition for the existence of a finite-order map of certain type is stated in the following result, which is an immediate consequence of Corollary 6.1.

Corollary 6.2 *Let p_1, p_2, p_3 and g be positive integers such that $g \geq 2$. There exists $f \in \mathcal{F}_g^+$ of type $[2g - 2 + p_1 + p_2 + p_3; 0; \{p_1, p_2, p_3\}]$ if and only if p_1, p_2, p_3 are pairwise coprime and p_i divides $2g - 2 + p_1 + p_2 + p_3$, for each $i \in \{1, 2, 3\}$.*

In particular, for each $g \geq 2$ and each k dividing g , there exist orientation-preserving finite-order maps of Σ_g of the following types:

$$[2g + k; 0; \{1, 1, k\}], [4g; 0; \{1, 1, 2g\}], [4g + 2; 0; \{1, 2, 2g + 1\}],$$

$$[2g; 0; \{1, 1, g\}], [2g + 1; 0; \{1, 1, 1\}], [2g + 2; 0; \{1, 1, 2\}].$$

We close this section with the following consequence of Theorem 5.12 and Lemma 5.15.

Lemma 6.3 *The types of the elements of \mathcal{F}_1^+ are precisely*

$$[2; 0; \{1, 1, 1, 1\}], [3; 0; \{1, 1, 1\}], [4; 0; \{1, 1, 2\}], [6; 0; \{1, 2, 3\}] \text{ and}$$

$$[n; 0; \{ \}], n \geq 1.$$

6.2 The orientation-reversing case

The aim of this section is to state and give a proof of the analogue of Corollary 6.1 for the orientation-reversing case.

We need to prove the analogue of Harvey's Theorem 5.14 for non-orientable planar discontinuous groups. In this case, we need to determine, not only the existence of an epimorphism $\phi: G \rightarrow \mathbb{Z}_n$ with kernel a surface group but also whether $\text{Ker}(\phi)$ is orientable. The following lemma states sufficient and necessary conditions for this.

Lemma 6.1 *Let G be a non-orientable planar discontinuous group and let $\phi: G \longrightarrow \mathbb{Z}_n$ be an epimorphism. Then $\text{Ker}(\phi)$ is orientable if and only if the following conditions hold:*

- (1) n is even;
- (2) For every generator x of G , $\varepsilon(x) = -1$ if and only if $\widetilde{\phi(x)}$ is odd.

Proof. Assume that $\text{Ker}(\phi)$ is orientable, so $\text{Ker}(\phi) \subset \text{Ker}(\varepsilon)$. Since ϕ and ε are surjective by hypothesis, there exists an epimorphism $\mathbb{Z}_n \mapsto \{-1, 1\}$ such that composed with ϕ it gives ε . Hence (1) holds. Since there is then a unique epimorphism $\mathbb{Z}_n \mapsto \{-1, 1\}$ and it sends odd elements to -1 and even elements to 1 , (2) holds.

Reverse reasoning proves the converse. ■

We now need to introduce some notation. If G is a planar discontinuous group of signature $(\vartheta, T, [m_1, m_2, \dots, m_R], B)$ and n is a positive integer divisible by each m_i , we write $\text{g.c.d}(G, n) = \text{g.c.d}(\frac{n}{m_1}, \frac{n}{m_2}, \dots, \frac{n}{m_R})$ and $p(G, n) = \frac{1}{2} \sum_{i=1}^R \frac{n}{m_i}$

Theorem 6.2 *Let n be a positive integer and let G be a non-orientable planar discontinuous group of signature*

$$(\vartheta, T, [m_1, m_2, \dots, m_R], B).$$

Then there exists an epimorphism $\phi: G \longrightarrow \mathbb{Z}_n$ such that $\text{Ker}(\phi)$ is an orientable surface group if and only if the following conditions hold:

- (1) n is even;
- (2) each m_i divides n ;
- (3) $\text{g.c.d}(G, n)$ is even;
- (4) If $B \geq 1$ or $p(G, n) \equiv_2 T + 1$ then $\frac{n}{2}$ is odd;
- (5) If $T + B = 1$ then $\text{g.c.d}(G, n) = 2$.

Proof. We prove the “only if” direction first. Let $\phi: G \longrightarrow \mathbb{Z}_n$ be an epimorphism such that $\text{Ker}(\phi)$ is an orientable surface group. By Lemma 6.1, (1) holds. By Lemma 5.10, $\phi(\sigma_i)$ has order m_i for each $i \in \{1, 2, \dots, R\}$, so (2) holds and

$$(6.1) \quad \widetilde{\phi(\sigma_i)} = \frac{k_i n}{m_i}$$

for some positive integer k_i such that $1 \leq k_i < m_i$, and $(k_i, m_i) = 1$. By Lemma 6.1, $\frac{k_i n}{m_i}$ is even. Clearly, if k_i is odd, then $\frac{n}{m_i}$ is even. If k_i is even, then m_i is odd, so by (1) $\frac{n}{m_i}$ is even. Hence, (3) holds.

Let us see (4). Assume first that $B \geq 1$. By Lemma 5.10, $\phi(\rho_{11}) = [\frac{n}{2}]_n$ and by Lemma 6.1, $\widetilde{\phi(\rho_{11})} = \frac{n}{2}$ is odd. So, we may assume that $B = 0$. Since G is non-orientable, $\vartheta = -$. Suppose now, that $\frac{n}{2}$ is even. Here, by (6.1), for each $i \in \{1, 2, \dots, R\}$, $\widetilde{\phi(\sigma_i)} \equiv_2 \frac{n}{2m_i}$. By Lemma 6.1, $\widetilde{\phi(\tau_i)} \equiv_2 1$ for each $i \in \{1, 2, \dots, T\}$. Then, since $\phi(\sigma_1 \sigma_2 \dots \sigma_R \tau_1^2 \tau_2^2 \dots \tau_R^2) = [0]_n$,

$$0 \equiv_2 \frac{1}{2} \sum_{i=1}^R \widetilde{\phi(\sigma_i)} + \sum_{i=1}^T \widetilde{\phi(\tau_i)} \equiv_2 p(G, n) + T$$

which proves (4).

To see (5), observe that, by (6.1), $\text{g.c.d}(G, n)$ divides $\sum_{i=1}^R \widetilde{\phi(\sigma_i)}$. Assume that $T + B = 1$. Then either $T = 1$, $B = 0$ and $\vartheta = -$ or $T = 0$, $B = 1$ and $\vartheta = +$. In the former case, since $\phi(\sigma_1 \sigma_2 \dots \sigma_R \tau_1^2) = [0]_n$, $\text{g.c.d}(G, n)$ divides $2\widetilde{\phi(\tau_1)}$. On the other hand, $\text{Im}(\phi)$ is generated by $[\text{g.c.d}(G, n)]_n$ and $\phi(\tau_1)$. Since ϕ is surjective, $\text{g.c.d}(\text{g.c.d}(G, n), \widetilde{\phi(\tau_1)}) = 1$. Then $\text{g.c.d}(G, n)$ divides 2 and since it is even the proof of this case is complete. In the latter case, that is, when $T = 0$, $B = 1$, and $\vartheta = +$, we have $\sum_{i=1}^R \phi(\sigma_i) + \phi(\pi_1) = [0]_n$. Then $\text{g.c.d}(G, n)$ divides $\widetilde{\phi(\pi_1)}$, so $\text{Im}(\phi)$ is generated by $[\text{g.c.d}(G, n)]_n$ and $\phi(\rho_{11}) = [\frac{n}{2}]_n$. Therefore, $\text{g.c.d}(\text{g.c.d}(G, n), \frac{n}{2}) = 1$. On the other hand, by (4), $\frac{n}{2}$ is odd, so, since $\text{g.c.d}(G, n)$ divides n , $\text{g.c.d}(\text{g.c.d}(G, n), \frac{n}{2}) = \frac{\text{g.c.d}(G, n)}{2}$. Thus, (5) holds.

We now prove the reverse implication. Consider first the case where $\vartheta = -$. If T is odd, define ϕ on the generators in the following way:

$$\begin{aligned} \phi(\sigma_i) &= [\frac{n}{m_i}]_n & \text{for } i \in \{1, 2, \dots, R\}, \\ \phi(\pi_i) &= [2]_n & \text{for } i \in \{1, 2, \dots, B\}, \\ \phi(\rho_{i1}) &= [\frac{n}{2}]_n & \text{for } i \in \{1, 2, \dots, B\}, \\ \phi(\tau_i) &= [(-1)^i]_n & \text{for } i \in \{2, 3, \dots, T\}, \\ \phi(\tau_1) &= \begin{cases} [-p(G, n) - 2B]_n & \text{if } p(G, n) \text{ is odd,} \\ [-p(G, n) - 2B + \frac{n}{2}]_n & \text{otherwise.} \end{cases} \end{aligned}$$

We claim that ϕ is surjective. If $B \geq 1$, by (4), $\frac{n}{2}$ is odd. Then $[1]_n \in \text{Im}(\phi)$ since $[2]_n, [\frac{n}{2}]_n \in \text{Im}(\phi)$ and $\text{g.c.d}(2, \frac{n}{2}) = 1$. If $B = 0$ and $T \geq 3$, ϕ is

surjective because $[1]_n \in \text{Im}(\phi)$ by definition. If $B = 0$ and $T = 1$, by (5) $\text{g.c.d}(G, n) = 2$. Then $[2]_n \in \text{Im}(\phi)$. On the other hand, by definition, $\widehat{\phi(\tau_1)}$ is odd. So, the claim follows directly.

Suppose now that T is even. Since $\vartheta = -$, $T \neq 0$. So, $T \geq 2$. In this case we define

$$\begin{aligned}\phi(\sigma_i) &= [\frac{n}{m_i}]_n & \text{for } i \in \{1, 2, \dots, R\}, \\ \phi(\pi_i) &= [0]_n & \text{for } i \in \{1, 2, \dots, B\}, \\ \phi(\rho_{i1}) &= [\frac{n}{2}]_n & \text{for } i \in \{1, 2, \dots, B\}, \\ \phi(\tau_i) &= [(-1)^i]_n & \text{for } i \in \{2, 3, \dots, T\}, \\ \phi(\tau_1) &= \begin{cases} [-\frac{p(G,n)}{2} - 1]_n & \text{for } \frac{p(G,n)}{2} \text{ is even,} \\ [-\frac{p(G,n)}{2} - 1 + \frac{n}{2}]_n & \text{otherwise.} \end{cases}\end{aligned}$$

Here, ϕ is surjective because $[1]_n \in \text{Im}(\phi)$.

If $\vartheta = +$ we define

$$\begin{aligned}\phi(\sigma_i) &= [\frac{n}{m_i}]_n & \text{for } i \in \{1, 2, \dots, R\}, \\ \phi(\tau_i) &= [2]_n & \text{for } i \in \{1, 2, \dots, T\}, \\ \phi(\rho_{i1}) &= [\frac{n}{2}]_n & \text{for } i \in \{1, 2, \dots, B\}, \\ \phi(\pi_1) &= [2 - 2B - 2p(G, n)]_n \\ \phi(\pi_i) &= [2]_n & \text{for } i \in \{2, 3, \dots, B\}.\end{aligned}$$

Notice that $B \geq 1$ because G is non-orientable. Therefore, by (4), $\frac{n}{2}$ is odd. Clearly, if $[2]_n, [\frac{n}{2}]_n \in \text{Im}(\phi)$, ϕ is surjective. It is trivial that this occurs when $B + T \geq 2$. Otherwise, since $B \geq 1$, $B + T = 1$. Then, by (5), $\text{g.c.d}(G, n) = 2$. Hence, in this case also, $[2]_n, [\frac{n}{2}]_n \in \text{Im}(\phi)$.

That ϕ is an orientable surface group follows from Lemma 6.1 and Lemma 5.10. ■

We now come to the main result of this section.

Corollary 6.3 *Let $B, R \geq 0$, $g \geq 2$, $n \geq 1$ and $1 \leq p_1, p_2, \dots, p_R < n$ be integers. Set*

$$T = 2 - R - B + \frac{2g - 2 + \sum_{i=1}^R p_i}{n}.$$

There exist $f \in \mathcal{F}_g^-$ of type $[n; B; \{p_1, p_2, \dots, p_R\}]$ if and only if the following conditions hold:

- (1) *For each $i \in \{1, 2, \dots, R\}$, p_i divides n ;*
- (2) *n is even;*

- (3) T is a non-negative integer;
- (4) If $B = 0$ then $T \geq 1$;
- (5) $\text{g.c.d}(p_1, p_2, \dots, p_R)$ is even;
- (6) If $B + T = 1$ then $\text{g.c.d}(p_1, p_2, \dots, p_R) = 2$;
- (7) If $B \geq 1$ or $\frac{1}{2} \sum_{i=1}^R p_i \equiv_2 T + 1$ then $\frac{n}{2}$ is odd.

Proof. Assume that there exists such an f . Clearly, (1) holds. By Theorem 5.12, there exists a non-orientable planar discontinuous group G with signature

$$\left(\vartheta, T, \left[\frac{n}{p_1}, \frac{n}{p_2}, \dots, \frac{n}{p_R} \right], B \right)$$

and an epimorphism $\phi: G \rightarrow \mathbb{Z}_n$ such that $\text{Ker}(\phi)$ is isomorphic to $\pi_1(\Sigma_g)$. Then (2), (3) and (4) hold, and (5), (6) and (7) follow from Theorem 6.2.

To see the converse, consider a non-Euclidean planar discontinuous group G with signature

$$\left(\vartheta, T, \left[\frac{n}{p_1}, \frac{n}{p_2}, \dots, \frac{n}{p_R} \right], B \right)$$

such that if $B = 0$ then $\vartheta = -$. Hence, G is non-orientable and we can apply Theorem 6.2 to conclude the existence of an epimorphism $\phi: G \rightarrow \mathbb{Z}_n$ such that $\text{Ker}(\phi)$ is an orientable surface group. By Theorem 5.12, there exists a finite-order orientation-reversing homeomorphism $f: \Sigma_g \rightarrow \Sigma_g$ of type $[n; B; \{p_1, p_2, \dots, p_R\}]$. Now, we complete the proof, by observing that, by Theorem 5.8, the signature of $\text{Ker}(\phi)$ is $(+, g, [], 0)$. ■

Conditions (2) and (5) of Corollary 6.3 implies the following.

Lemma 6.4 *If $\mathcal{F}_{g,b}^- \neq \emptyset$ then b is even.*

An application of Corollary 6.3 yields the following result, which will be used in the proof of Theorem C.

Lemma 6.5 *Let p, p_1, p_2 and g be positive integers such that $g \geq 2$.*

- (1) *There exist $f \in \mathcal{F}_g^-$ of type $[2g - 2 + p; 0; \{p\}]$ if and only if g is odd, p is even, and p divides $2g - 2$.*
- (2) *There exist $f \in \mathcal{F}_g^-$ of type $[2g - 2 + p_1 + p_2; 0; \{p_1, p_2\}]$ if and only if g is even, $\text{g.c.d}(p_1, p_2) = 2$, and, for each $i \in \{1, 2\}$, p_i divides $2g - 2 + p_1 + p_2$.*

Proof. The reverse implication in (1) as well as the reverse implication in (2) follow easily from Corollary 6.3. So, it only remains to prove both direct implications.

We start with (1). Set $T = 2 - 1 - 0 + \frac{2g-2+p}{2g-2+p} = 2$. Assume that there exists $f \in \mathcal{F}_g^-$ of type $[2g - 2 + p; 0; \{p\}]$. From Corollary 6.3(5), it follows that p is even. By Corollary 6.3(1), p divides $2g - 2 + p$, so, p divides $2g - 2$. If $\frac{p}{2}$ is even, since $\frac{p}{2}$ divides $g - 1$ we have that g is odd. If $\frac{p}{2}$ is odd, $\frac{p}{2} \equiv_2 T + 1 = 3$. Hence by Corollary 6.3(7), $\frac{n}{2} = g - 1 + p$ is odd. So, g is odd.

Now, we prove (2). Set $T = 2 - 2 - 0 + \frac{2g-2+p_1+p_2}{2g-2+p_1+p_2} = 1$. Assume that there exists $f \in \mathcal{F}_g^-$ of type

$$[2g - 2 + p_1 + p_2; 0; \{p_1, p_2\}].$$

By Corollary 6.3(1), p_i divides $2g - 2 + p_1 + p_2$ for each $i \in \{1, 2\}$. Since $B + T = 0 + 1 = 1$, by Corollary 6.3(6), $\text{g.c.d}(p_1, p_2) = 2$. Then, both $\frac{p_1}{2}, \frac{p_2}{2}$ are odd or one of them is even and the other odd. In the former case, we have that

$$\frac{p_1 + p_2}{2} \equiv_2 0 \equiv_2 T + 1,$$

and by Corollary 6.3(7), $\frac{n}{2} = g - 1 + \frac{p_1+p_2}{2}$ is odd. So, g is even. To study the latter case we can assume without loss of generality that $\frac{p_1}{2}$ is even and $\frac{p_2}{2}$ is odd. Since p_1 divides $2g - 2 + p_1 + p_2$, $\frac{p_1}{2}$ divides $g - 1 + \frac{p_2}{2}$. Hence, $g - 1$ is odd. Therefore, g is even and the proof is complete. ■

6.3 Examples

The purpose of this section is to construct finite-order orientation-reversing homeomorphisms of closed surfaces. We shall do this by “gluing together” finite-order orientation-preserving ones. Some of the ideas of these constructions are based on an example given in [39]. These finite-order maps will be used in Chapter 9 to construct maps with a given minimum period.

Before proving these result, let us introduce some notation. For each $\alpha \in \mathbb{R}$, define $R_\alpha: \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ by $z \mapsto ze^{2\pi\alpha i}$.

Lemma 6.1 *Let g be a positive even integer. Then there exists $f \in \mathcal{F}_g^-$ of type*

$$[4g + 4; 0; \{4, 2g + 2\}],$$

and a closed annulus $A \subset \Sigma_g$ such that $f|_A$ is conjugate to the map $(z, t) \mapsto (R_{\frac{1}{4g+4}}(z), 1 - t)$ on $\mathbb{S}^1 \times [0, 1]$.

Proof. Set $l = \frac{g}{2}$. By Lemma 6.3 and Corollary 6.2, there exists $h \in \mathcal{H}_l^+$ of type $[4l + 2; 0; \{1, 2, 2l + 1\}]$. We can assume without loss of generality that $\Sigma_l \subset \mathbb{R}^3$. Let D be an open disk centered at the fixed point of h such that $h(D) = D$, and let R be a plane which does not intersect Σ_l . Denote by $s_R: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ the reflection with respect to this plane. Now define a map

$$k: (\Sigma_l \setminus D) \cup s_R(\Sigma_l \setminus D) \longrightarrow (\Sigma_l \setminus D) \cup s_R(\Sigma_l \setminus D)$$

in the following way:

$$k(x) = \begin{cases} s_R(x) & \text{if } x \in \Sigma_l \setminus D, \\ h(s_R(x)) & \text{if } x \in s_R(\Sigma_l \setminus D). \end{cases}$$

Since h has order $4l + 2$, there exists q coprime to $4l + 2$ and a parametrization $\omega: \mathbb{S}^1 \rightarrow \partial D$ such that $\omega^{-1} \circ h \circ \omega = R_{\frac{q}{4l+2}}$. By taking a power of h if necessary, we can assume that $q = 1$. Now define an equivalence relation \sim on $(\Sigma_l \setminus D) \cup s_R(\Sigma_l \setminus D)$ as follows: Let $x, y \in (\Sigma_l \setminus D) \cup s_R(\Sigma_l \setminus D)$. Then $x \sim y$ if and only if one of the following statements holds:

- (1) $x = y$;
- (2) $x \in \partial D$, $y \in s_R(\partial D)$, $x = \omega(z)$ and $y = s_P(\omega(R_{-\frac{1}{8l+4}}(z)))$;
- (3) $y \in \partial D$, $x \in s_R(\partial D)$, $y = \omega(z)$ and $x = s_P(\omega(R_{-\frac{1}{8l+4}}(z)))$.

Observe that

$$k(s_P(\omega(R_{-\frac{1}{8l+4}}(z)))) = \omega(R_{\frac{1}{8l+4}}(z)) \sim s_R(\omega(z)) = k(\omega(z)).$$

From this we can conclude that $x \sim y$ implies $k(x) \sim k(y)$. On the other hand, $(\Sigma_l \setminus D) \cup s_R(\Sigma_l \setminus D) / \sim$ is a closed surface of genus g . Hence, k induces $f \in \mathcal{F}_g^-$. Clearly, the type of k is $[4g + 4; 0; \{4, 2g + 2\}]$.

Denoting by $p: (\Sigma_l \setminus D) \cup s_R(\Sigma_l \setminus D) \rightarrow \Sigma_g$ the natural projection, it is not hard to see that there exists an annulus $A \subset \Sigma_g$ invariant under f such that $p(\partial D) \subset A$ and $f|_A$ is conjugate to the map $(z, t) \mapsto (R_{\frac{1}{4g+4}}(z), 1 - t)$ on $\mathbb{S}^1 \times [0, 1]$. ■

Lemma 6.2 *Let g be a positive odd integer such that $g \geq 3$. Then there exists $f \in \mathcal{F}_g^-$ of type*

$$[4g - 4; 0; \{2g - 2\}],$$

and a closed annulus $A \subset \Sigma_g$ such that $f|_A$ is conjugate to the map $(z, t) \mapsto (R_{\frac{1}{4g-4}}(z), 1-t)$ on $\mathbb{S}^1 \times [0, 1]$.

Proof. By Corollary 6.3, there exists $h \in \mathcal{F}_{g-1}^r$ of type

$$[4g-4; 0; \{2, 2g-2\}]:$$

Let D_1, D_2 be open disks centered at points of the 2-periodic orbit of h such that $h(D_1) = D_2$. Set

$$\eta = h|_{\Sigma_{g-1} \setminus (D_1 \cup D_2)}.$$

Taking a power of η if necessary, we can assume that $\eta^2|_{\partial D_1}$ is conjugate to rotation through an angle of $\frac{2\pi}{4g-4}$. Then there exists a parametrization $\omega: \mathbb{S}^1 \rightarrow \partial D_1$ such that $\omega^{-1}\eta^2\omega = R_{\frac{1}{4g-4}}$.

Let \sim be the smallest equivalence relation defined on $\Sigma_{g-1} \setminus (D_1 \cup D_2)$ containing all the pairs of the form

$$(x, \omega R_{\frac{-1}{8g-8}} \omega^{-1} \eta(x))$$

for each $x \in \partial D_2$. Set $y = \omega R_{\frac{-1}{8g-8}} \omega^{-1} \eta(x)$. Since $y \in \partial D_1$, $\eta(y) \in \partial D_2$. Therefore,

$$\eta(y) \sim \omega R_{\frac{-1}{8g-8}} \omega^{-1} (\eta^2(y)) = \omega R_{\frac{-1}{8g-8}} \omega^{-1} (y) = \eta(x).$$

From this it follows that $x \sim y$ implies that $\eta(x) \sim \eta(y)$ for every $x, y \in \Sigma_{g-1} \setminus (D_1 \cup D_2)$. On the other hand, $\Sigma_{g-1} \setminus (D_1 \cup D_2) / \sim$ is an orientable closed surface of genus g . It is easy to check that η induces an element of \mathcal{F}_g^- with the required properties. \blacksquare

Lemma 6.3 *Let g be a positive even integer. Then there exists $f \in \mathcal{H}_g^-$ of type*

$$[2g-2; 0; \{ \}],$$

and a closed annulus $A \subset \Sigma_g$ such that $f|_A$ is conjugate to the map $(z, t) \mapsto (R_{\frac{1}{2g-2}}(z), 1-t)$ on $\mathbb{S}^1 \times [0, 1]$.

Proof. By Corollary 6.3 there exists $h \in \mathcal{F}_{g-1}^-$ of type $[2g-2; 0; \{2\}]$. Let D_1, D_2 be open disks centered at points of the 2-periodic orbit of h such that $h(D_1) = D_2$. Now we can proceed as in Lemma 6.2, considering $S = \Sigma_{g-1} \setminus (D_1 \cup D_2)$, $\eta = h|_S$, “gluing” the boundary components in an appropriate way and taking the map induced by η in the quotient surface. \blacksquare

Lemma 6.4 *Let g be a positive integer even. Then there exists $f \in \mathcal{F}_g^-$ of type $[4g; 0; \{2, 2g\}]$ and a closed annulus $A \subset \Sigma_g$ such that $f|_A$ is conjugate to the map $(z, t) \mapsto (R_{\frac{1}{4g}}(z), 1 - t)$ on $\mathbb{S}^1 \times [0, 1]$.*

Proof. Set $k = \frac{g}{2}$. By Lemma 6.3 and Corollary 6.2 there exists $h \in \mathcal{F}_k^+$ of type $[4k; 0; \{1, 1, 2k\}]$. By taking an h -invariant disk around one of the fixed points of h , it is easy to check that we can complete the proof as in Lemma 6.1. ■

Remark 6.5 Dropping the requirement of the existence of the invariant annulus A in Lemmas 6.1, 6.2, 6.3 and 6.4, these results follow directly from Corollaries 6.1 and 6.3. Nevertheless, the existence of such an annulus will be fundamental for the construction of examples, as we will see in Chapter 9 □

Chapter 7

Consequences of fixed-point theory

This chapter is devoted to stating some facts concerning the Lefschetz numbers of homeomorphisms of surfaces. In Section 7.1 we prove a result which contains one of the main ideas of this thesis, that is, a relationship between homeomorphisms of surfaces with boundary and homeomorphisms of surfaces without boundary. Some algebraic tools are described in Section 7.2. In Section 7.3 we prove some results about sequences of Lefschetz numbers which will be used in Chapter 9 and 10. Finally, Section 7.4 is devoted to the (tedious) study of three particular classes of homeomorphisms of closed surfaces which will be used in Chapter 11.

7.1 The induced map

We begin this section by introducing a definition which we will use frequently. Let $f \in \mathcal{H}_{g,b}$ and let T be an f -invariant subset of $\Sigma_{g,b}$ consisting of k boundary components of $\Sigma_{g,b}$. In $\Sigma_{g,b}$ we consider the equivalence relation where two points in $\Sigma_{g,b}$ are equivalent if they are equal or they belong to the same boundary component in T . The quotient space is a surface of genus g with $b - k$ boundary components. Since T is f -invariant, f induces a homeomorphism $\tilde{f}^T \in \mathcal{H}_{g,b-k}$. In particular, we can take T to be the boundary of $\Sigma_{g,b}$. In this case $b = k$, so f induces a homeomorphism of Σ_g which will be denoted by \tilde{f} and called the *homeomorphism induced by f* .

The aim of this section is to prove a topological result, Proposition 7.2,

which will be fundamental for (almost all) our arguments and states a relationship between the fixed points of a map $f: \Sigma_{g,b} \longrightarrow \Sigma_{g,b}$ and the index of the fixed points of the induced map. In this way, we will be able to obtain information on the fixed-point set of maps of surfaces with boundary by studying the (simpler) case of maps of surfaces without boundary.

Lemma 7.1 *Let $f \in \mathcal{H}_{g,b}$. Assume that there exists an f -invariant boundary component B of $\Sigma_{g,b}$. Set $T = \{B\}$ and consider*

$$\tilde{f}^T: \Sigma_{g,b-1} \longrightarrow \Sigma_{g,b-1},$$

the map induced by f on $\Sigma_{g,b-1}$ by collapsing B to a point, q . If $f|_B$ is fixed-point free then $\text{Ind}_{\tilde{f}^T}(q) = 1$.

Proof. Suppose $f|_B$ is fixed-point free. Here, there exists an open set $U \subset \Sigma_{g,b}$ such that $B \subset U$ and $f|_U$ is also fixed-point free. Set $V = \Sigma_{g,b} \setminus B$. Clearly, V is an open set, $V \cup U = \Sigma_{g,b}$ and $U \cap V \cap \text{Fix}(f) = \emptyset$. Hence, we can apply Theorem 2.7 to conclude that

$$L(f) = I(f|_U) + I(f|_V).$$

Since $f|_U$ is fixed-point free, by Remark 2.3, $I(f|_U) = 0$, so

$$(7.1) \quad L(f) = I(f|_V).$$

Let p denote the projection from $\Sigma_{g,b}$ to the quotient space $\Sigma_{g,b-1}$. Then $p(U)$ and $p(V)$ are open sets contained in $\Sigma_{g,b-1}$ and satisfy $p(U) \cup p(V) = \Sigma_{g,b-1}$, $p(U) \cap p(V) \cap \text{Fix}(\tilde{f}^T) = \emptyset$. Thus, by Theorem 2.7,

$$(7.2) \quad L(\tilde{f}^T) = I(\tilde{f}^T|_{p(U)}) + I(\tilde{f}^T|_{p(V)}).$$

On the other hand, setting $A = B$ in Proposition 2.10 we obtain

$$L(f) + 1 = L(\tilde{f}^T) + L(f|_B).$$

Since $f|_B$ is fixed-point free, by Theorem 2.2, $L(f|_B) = 0$. By (7.1) and (7.2) we get

$$(7.3) \quad I(f|_V) + 1 = I(\tilde{f}^T|_{p(U)}) + I(\tilde{f}^T|_{p(V)}).$$

From the definition of \tilde{f}^T it follows that $I(\tilde{f}^T|_{p(V)}) = I(f|_V)$ so, (7.3) becomes

$$I(\tilde{f}^T|_{p(U)}) = 1.$$

Clearly $\text{Ind}_{\tilde{f}^T}(q) = I(\tilde{f}^T|_{p(V)})$, so our proof is complete. ■

Proposition 7.2 *Let $f \in \mathcal{H}_{g,b}$ and let $\tilde{f}: \Sigma_g \rightarrow \Sigma_g$ be the map induced by f . If, for some i , f^i is fixed-point free, then all the fixed points of \tilde{f}^i are isolated and have local index equal to 1 with respect to \tilde{f}^i .*

Proof. Let x be a fixed point of \tilde{f}^i . Since f^i is fixed-point free, x is a collapsed boundary component. Since the set of collapsed boundary components is finite, x is isolated. That the index is 1 follows from Lemma 7.1. ■

7.2 Basic algebraic tools

The goal of this section is to present and prove some elementary algebraic facts concerning the map f_{*1} induced by a map $f: \Sigma \rightarrow \Sigma$. The reason for doing this is that by (2.3) there exists a close relationship between the map f_{*1} and the sequence of Lefschetz numbers, $L(f), L(f^2), \dots$.

Recall that, by Proposition 2.2, there exists a basis such that for each $f \in \mathcal{H}_g$, the matrix representing f_{*1} is symplectic. In the following lemma we state a property of such matrices. The proof of the second statement is due to Fenchel and can be found in [35], but we include it here for convenience. A proof of this statement for the improper symplectic case can also be found in [14].

Lemma 7.1 *Suppose $M \in GL_{2g}(\mathbb{Z})$ is proper (resp. improper) symplectic and let $P(x) = \det(x \text{Id}_{2g} - M)$ be its characteristic polynomial. Then*

- (1) $\det(M) = 1$ (resp. $\det(M) = (-1)^g$).
- (2) $P(x) = x^{2g}P(1/x)$ (resp. $P(x) = (-1)^g x^{2g}P(-1/x)$).

Proof. If M is proper symplectic then (1) holds by Proposition 2.1. Now, define $U_g \in GL_{2g}(\mathbb{Z})$ as

$$\begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}.$$

It is not hard to see that U_g is improper symplectic and $\det(U_g) = (-1)^g$. Let $M \in GL_{2g}(\mathbb{Z})$ be an improper symplectic matrix. Since $U_g M$ is proper symplectic, then, by Proposition 2.1, $(-1)^g \det(U_g) = \det(MU_g) = 1$. Hence, (1) holds.

To see (2), assume that $M \in GL_{2g}(\mathbb{Z})$ is proper symplectic. (The argument where M is improper symplectic is similar.) Then,

$$M^t J_g (\lambda I_{2g} - M) = \lambda M^t J_g - M^t J_g M = (\lambda M - I_{2g})^t J_g$$

$$= -\lambda I_{2g} \left(\frac{1}{\lambda} I_{2g} - M \right)^t J_g,$$

and (2) follows by taking determinants of the two extremes of the above chain of equalities. ■

We need to study the sequence of integers

$$L(f), L(f^2), \dots$$

which, by (2.3), is equivalent to studying the sequence

$$p_1, p_2, \dots$$

By Newton's equations (page 22), this infinite sequence is closely related to the sequence

$$s_1, s_2, \dots, s_{2g}.$$

For this reason, in the next result, which follows directly from Proposition 2.2 and Lemma 7.1, we state some relations satisfied by the s_i 's.

Lemma 7.2 *If $f \in \mathcal{H}_g^+$ then $\det(f_{*1}) = 1$ and $s_h = s_{2g-h}$ for all $h \in \{1, \dots, 2g\}$. If $f \in \mathcal{H}_g^-$ then $\det(f_{*1}) = (-1)^g$ and $s_h = (-1)^{g+h} s_{2g-h}$ for all $h \in \{1, \dots, 2g\}$.*

Remark 7.3 By Lemma 7.2 and Newton's equations (page 22), for each $f \in \mathcal{H}_g$, the infinite sequence $L(f), L(f^2), \dots$ is determined by the finite sequence $L(f), L(f^2), \dots, L(f^g)$. In other words, if we are given

$$L(f), L(f^2), \dots, L(f^g)$$

then we can compute $L(f^i)$ for each positive integer i . Hence, if $f, h \in \mathcal{H}_g^+$ (resp. $f, h \in \mathcal{H}_g^-$) and $L(f^i) = L(h^i)$ for each $i \in \{1, 2, \dots, g\}$ then $L(f^i) = L(h^i)$ for each positive integer i . □

The arguments to prove the next result, from which Proposition A follows as a corollary, are based on Nielsen [35].

Lemma 7.4 *Let $A \in GL_k(\mathbb{R})$ where $k \geq 3$. If $\text{trace}(A^i) = 1$ for $i \in \{-1, 1, 2, \dots, k-2\}$ then $\text{trace}(A^{k-1}) \neq 1$.*

Proof. The assumptions imply that $p_i = 1$ for $i = 1, 2, \dots, k-2$. Then by Newton's equations (page 22),

$$\begin{aligned}
1 + s_1 &= 0, \\
1 + s_1 + 2s_2 &= 0, \\
1 + s_1 + s_2 + 3s_3 &= 0, \\
&\dots \\
1 + s_1 + s_2 + s_3 + \dots + (k-2)s_{k-2} &= 0, \\
p_{k-1} + s_1 + s_2 + s_3 + \dots + (k-1)s_{k-1} &= 0.
\end{aligned}$$

Hence, $s_1 = -1$, $s_i = 0$ for each $i \in \{2, 3, \dots, k-2\}$ and $(k-1)s_{k-1} = 1 - p_{k-1}$. Therefore, $p_{k-1} = 1$ if and only if $s_{k-1} = 0$. On the other hand, since the eigenvalues of A are the inverses of the eigenvalues of A^{-1} ,

$$s_{k-1} = (-1)^{k-1} \det(A) \operatorname{trace}(A^{-1}) = (-1)^{k-1} \det(A) \neq 0.$$

Therefore, $p_{k-1} \neq 1$ and the proof is complete. \blacksquare

To prove our main results we need to prove the existence of fixed-point classes of f^m with negative index, for certain m . In view of Lemma 2.9, one way to do this is to show that $L(f^m) < 0$. For this reason we are going to study some conditions on f which imply that inequality.

Lemma 7.5 *Suppose that $g \geq 2$ and that $f: \Sigma_g \rightarrow \Sigma_g$ is a homeomorphism such that, for some positive integer m , the m -th power of each of the eigenvalues of f_{*1} is equal to 1. Then $L(f^m) < 0$.*

Proof. By hypothesis, $\operatorname{trace}(f_{*1}^m) = \sum_{i=1}^{2g} \lambda_i^m = 2g$. Then, by (2.2), if f^m preserves orientation then $L(f^m) = 2 - \operatorname{trace}(f_{*1}^m) = 2 - 2g < 0$ and if f^m reverses orientation then $L(f^m) = -\operatorname{trace}(f_{*1}^m) = -2g < 0$. \blacksquare

7.3 Sequences of Lefschetz numbers I

The results of this section, as well as the ones of the next, are consequences of (2.3), Newton's equations (page 22) and Lemma 7.2.

Lemma 7.1 *Let $f \in \mathcal{H}_g^+$ be such that $L(f^i) = 1$ for each $i \in \{1, 2, \dots, g\}$. Then $L(f^{g+1}) = -g$.*

Proof. By the assumptions and (2.3), $p_i = 1$ for each $i \in \{1, 2, \dots, g\}$, and, by Newton's equations (N.1), (N.2) (page 22), \dots , (N.g), $s_1 = -1$ and $s_i = 0$ for each $i \in \{2, 3, \dots, g\}$. By Lemma 7.2, $s_{g+1} = s_1 = -1$. Hence, we can replace all these values in Newton's equation (N.(g+1)) (page 22) to obtain $p_{g+1} = g + 2$. Thus, by (2.3), $L(f^{g+1}) = 2 - p_{g+1} = -g$, as desired. ■

Lemma 7.2 *If $g \geq 2$ then there does not exist $f \in \mathcal{H}_g^+$ such that $L(f^i) \geq 4$ for each $i \in \{1, 2, \dots, 2g\}$.*

Proof. Assume that there does exist such an f . By (2.3), $p_i = 2 - L(f^i) < -1$, for each $i \in \{1, 2, \dots, 2g\}$. We claim that $s_i > 1$ for each $i \in \{1, 2, \dots, 2g\}$. We prove this claim by induction. By Newton's equation (N.1) (page 22), $s_1 = -p_1 > 1$. Hence, the claim holds for $i = 1$. Now, assume $j \leq 2g$ is such that $s_i > 1$ for each $i \in \{1, 2, \dots, j-1\}$. Observe that $p_i s_{j-i} < -1$ for each $i \in \{1, 2, \dots, j-1\}$. By Newton's equations (page 22),

$$-j s_j = p_1 s_{j-1} + p_2 s_{j-2} + \dots + p_{j-1} s_1 + p_j < -j,$$

so the claim holds for j and we are done. In particular, we have proved that $s_{2g} > 1$. On the other hand, by Lemma 7.2, $s_{2g} = \det(f_{*1}) = 1$, which is impossible. ■

Lemma 7.3 *If $g \geq 2$ then there does not exist $f \in \mathcal{H}_g^-$ such that $L(f^{2i}) > 4$ and $L(f^{2i-1}) = 0$ for each $i \in \{1, 2, \dots, g\}$.*

Proof. Assume that there does exist such an f . Let $i \in \{1, 2, \dots, 2g\}$. By (2.3),

$$(7.4) \quad p_{2i} = 2 - L(f^{2i}) < -2 \text{ and } p_{2i-1} = -L(f^{2i-1}) = 0.$$

We claim that $s_i > 1$ if i is even and $s_i = 0$ if i is odd. We prove this claim by induction. By Newton's equation (N.1) (page 22), $s_1 = -p_1 = 0$. Hence, the claim holds for $i = 1$. Now, assume $j \leq 2g$ is such that the inductive hypothesis holds for each $i \in \{1, 2, \dots, j-1\}$. Suppose that j is even. Observe that by the inductive hypothesis and (7.4), $p_i s_{j-i} < -2$ for each even $i \in \{1, 2, \dots, j-1\}$. By Newton's equations (page 22) and (7.4)

$$-j s_j = p_1 s_{j-1} + p_2 s_{j-2} + \dots + p_{j-1} s_1 + p_j < -j,$$

so the claim holds in this case. If j is odd, for each $i \in \{1, 2, \dots, j\}$, $p_i s_{j-1} = 0$ because, by the inductive hypothesis and (7.4), either $p_i = 0$ or $s_{j-i} = 0$. Hence,

$$-j s_j = p_1 s_{j-1} + p_2 s_{j-2} + \dots + p_{j-1} s_1 + p_j = 0,$$

so the claim is proved. In particular, we have proved that $s_{2g} > 1$. On the other hand, by Lemma 7.2, $s_{2g} = \det(f_{*1}) = (-1)^g$, which is impossible. ■

7.4 Sequences of Lefschetz numbers II

In Section 11 we will need to prove the existence of fixed-point classes of negative index for pseudo-Anosov maps in standard form. As we will see, the pseudo-Anosov maps with more than two orbits of singularities can be “controlled” with the help of the Euler-Poincaré Formula (3.2). Some of the homeomorphisms with one or two orbits of singularities do not offer many difficulties and can be studied by using the Euler-Poincaré Formula or Lemma 7.5. However, there are some particular cases of pseudo-Anosov homeomorphisms with one or two orbits of singularities which need special treatment. The results of this section will allow us to deal with these cases.

7.4.1 The orientation-preserving case

We start with two technical lemmas. The first one will be used to prove the second one which, in turn, will be used in the proof of Proposition 7.3.

Lemma 7.1 *Let $A \in GL_k(\mathbb{Z})$ be such that, for some positive integer n and some non-negative integer j , the coefficients s_i of the characteristic polynomial of A satisfy*

$$s_i = \begin{cases} -2 & \text{if } i \equiv_n 1, \\ 1 & \text{if } i \equiv_n 0, 2, \\ 0 & \text{otherwise,} \end{cases}$$

for non-negative i such that $i \leq nj$. Then, for each positive integer t such that $nj + t \leq k$

$$p_{nj+t} + \sum_{h=0}^{j-1} p_{nh+t} - 2 \sum_{h=1}^j p_{nh+t-1} + \sum_{h=1}^j p_{nh+t-2}$$

$$+ \sum_{h=1}^{t-1} s_{nj+h} p_{t-h} + (nj+t)s_{nj+t} = 0.$$

Proof. Since $nj+t \leq k$, the result follows directly from Newton's equation N.(nj+t) (page 22)

$$(7.5) \quad p_{nj+t} + s_1 p_{nj+t-1} + s_2 p_{nj+t-2} + \dots + (nj+t)s_{nj+t} = 0,$$

by replacing the s_h 's by their values for $h \in \{1, 2, \dots, nj\}$ and grouping the p_h 's for $h \in \{t, t+1, \dots, nj+t-1\}$ according to the value of h modulo n . ■

Lemma 7.2 *Let $g \geq 2$, let $f \in \mathcal{H}_g^+$, let v, n and i be positive integers such that $v \leq 2g$ and $n \geq 3$. If for each $i \leq v$*

$$L(f^i) = \begin{cases} n & \text{if } n \text{ divides } i, \\ 0 & \text{otherwise,} \end{cases}$$

then, for each $i \leq v$

$$s_i = \begin{cases} -2 & \text{if } i \equiv_n 1, \\ 1 & \text{if } i \equiv_n 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, if j, t are positive integers such that $nj \leq v$ and $nj+t \leq 2g$ then

$$\begin{aligned} p_{nj+t} + \sum_{h=0}^{j-1} p_{nh+t} - 2 \sum_{h=1}^j p_{nh+t-1} + \sum_{h=1}^j p_{nh+t-2} \\ + \sum_{h=1}^{t-1} s_{nj+h} p_{t-h} + (nj+t)s_{nj+t} = 0. \end{aligned}$$

Proof. Assume that we have proved the first statement of the lemma. Then the second one follows immediately by Lemma 7.1. So, let us prove the formula for the s_i 's.

By (2.3),

$$p_i = \begin{cases} 2-n & \text{if } n \text{ divides } i, \\ 2 & \text{otherwise,} \end{cases}$$

for $i \leq v$. We shall proceed by induction. By Newton's equation (N.1) (page 22), the equality holds for $i = 1$. Consider now $1 \leq i \leq v - 1$ and suppose that the statement holds for s_1, s_2, \dots, s_i . Let j and r be the non-negative integers such that $i = nj + r$ and $0 \leq r < n$. Recall that a matrix representing f_{*1} has size $2g \times 2g$ and observe that $2 \leq i + 1 = nj + r + 1 \leq v \leq 2g$. By the induction hypothesis and Lemma 7.1 with $t = p + 1$,

$$(7.6) \quad p_{i+1} + \sum_{h=0}^{j-1} p_{nh+r+1} - 2 \sum_{h=1}^j p_{nh+r} + \sum_{h=1}^j p_{nh+r-1} + s_{nj+1}p_r + s_{nj+2}p_{r-1} + \dots + s_{nj+r}p_1 + (i+1)s_{i+1} = 0$$

Considering separately the four cases, $r = 0, 1, n-1$ and $r \notin \{0, 1, n-1\}$, we show that the values of the p_l 's for $l \in \{1, 2, \dots, i+1\}$ and s_l 's for $l \in \{nj+1, nj+2, \dots, nj+r\}$ can be replaced to find s_{i+1} . Indeed, if $r = 0$, then (7.6) can be written as

$$2 + 2j - 2(2-n)j + 2j + (-2)(2-n) + 1 \cdot 2 + 0 + \dots + (nj+1)s_{i+1} = 0,$$

which gives $s_{i+1} = -2$.

If $r = 1$ then (7.6) gives

$$2 + 2j - 4j + (2-n)j - 4 + (nj+2)s_{i+1} = 0,$$

so $s_{i+1} = 1$.

If $r = n-1$ then (7.6) becomes

$$2 - n + (2-n)j - 2 \cdot 2j + 2j - 2 \cdot 2 + 2 + (nj+n)s_{i+1} = 0,$$

and hence, $s_{i+1} = 1$.

Finally, suppose that $r \notin \{0, 1, n-1\}$. Then, by (7.6),

$$2 + 2j - 4j + 2j - 2 \cdot 2 + 2 + (i+1)s_{i+1} = 0,$$

which gives $s_{i+1} = 0$. ■

The following result will be used to prove Proposition 11.2.

Proposition 7.3 *Let $g \geq 2$ and let $f \in \mathcal{H}_g^+$ satisfy the following conditions:*

(1) $1, 2, g + 1 \notin \text{Per}(f)$.

(2) *There exists a positive integer n such that $\text{Per}(f) \cap \{3, 4, \dots, g\} = \{n\}$. Moreover, there is only one periodic orbit of period n and each of its points has index 1 for f^{hn} for each positive integer h such that $hn \leq g$.*

Then either there exists m such that $1 \leq m \leq 3g - 3$ and $L(f^m) < 0$, or there exists $l \in \text{Per}(f)$ such that $g + 2 \leq l \leq \frac{4}{3}(g - 1)$

Proof. Here there are only finitely many periodic points of f of period less than or equal to $2g$, so they are isolated. Therefore, by Lemma 2.7, if $1 \leq i \leq 2g$, then the Lefschetz number of f^i is the sum of the local indices of its fixed points, so, for each $i \leq g + 1$,

$$L(f^i) = \begin{cases} n & \text{if } n \text{ divides } i, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, by (2.3),

$$(7.7) \quad p_i = \begin{cases} 2 - n & \text{if } n \text{ divides } i, \\ 2 & \text{otherwise,} \end{cases}$$

for $i \leq g + 1$.

Let r, j be the non-negative integers such that $g = nj + r$ and $0 \leq r \leq n - 1$. We claim that $r \notin \{0, 2\}$. Since $n \geq 3$, by Lemmas 7.2 and 7.2, if $r = 0$, $-2 = s_{g+1} = s_{g-1} \in \{0, 1\}$, which is a contradiction. Similarly, if $r = 2$ then $-2 = s_{g-1} = s_{g+1} \in \{0, 1\}$. This completes the proof of the claim. Now we split the proof into two cases.

Case 1. $r = 1$.

By Lemmas 7.2 and 7.2,

$$P(x) = (x - 1)^2(x^{2nj} + x^{n(2j-1)} + \dots + x^n + 1) = \frac{x^{2nj+n} - 1}{x^n - 1}(x - 1)^2.$$

Then $(x^n - 1)P(x) = (x^{2nj+n} - 1)(x - 1)^2$. Hence, if $\lambda \neq 1$ is a root of $P(x)$ then $0 = (\lambda^n - 1)P(\lambda) = (\lambda^{2nj+n} - 1)(\lambda - 1)^2$. Therefore the $(2nj + n)$ -th power of the roots of $P(x)$ are 1. Since these roots are the eigenvalues of f_{*1} , Lemma 7.5 completes the proof of this case because $2nj + n \leq 2g - 2 + g - 1 \leq 3g - 3$.

Case 2. $r \geq 3$.

We start by showing by induction on t that if $r + 1 \leq t < \min\{n, 2r - 2\}$ then $p_{nj+t} = 2$. Since $nj + r + 1 = g + 1$ and $0 \leq r + 1 \leq t < n$, n does not divide $g + 1$, so, by (7.7), the result holds for $t = r + 1$. Now fix $r + 1 < t < \min\{n, 2r - 2\}$ and assume that $p_{nj+s} = 2$ for every $r + 1 \leq s < t$.

Recall that a matrix representing f_{*1} has size $2g \times 2g$ and observe that $1 \leq nj + t < nj + n < 2g$. Then, by Lemma 7.2 (taking v as $g + 1$),

$$(7.8) \quad \begin{aligned} & p_{nj+t} + \sum_{h=0}^{j-1} p_{nh+t} - 2 \sum_{h=1}^j p_{nh+t-1} + \sum_{h=1}^j p_{nh+t-2} \\ & + \sum_{h=1}^{t-1} s_{nj+h} p_{t-h} + (nj + t) s_{nj+t} = 0. \end{aligned}$$

Since $4 \leq r + 1 \leq t < n$, $t - 2, t - 1, t \not\equiv_n 0$. By Lemmas 7.2 and 7.2, $s_{nj+1} = -2$, $s_{nj+2} = 1$ and $s_{nj+s} = 0$ if $0 \leq s < t$. Further, our induction hypothesis implies that $p_{nj+s} = 2$ if $r + 1 \leq s < t$. Now, substituting in (7.8) these values and the ones given by (7.7), we obtain

$$(7.9) \quad p_{nj+t} + 2j - 4j + 2j - 2 \cdot 2 + 2 + (nj + t) s_{nj+t} = 0.$$

Since $r + 1 \leq t < 2r - 2$, we have $2 < 2r - t < r$. Therefore, by Lemmas 7.2 and 7.2, $s_{nj+t} = s_{2g-(nj+t)} = s_{nj+2r-t} = 0$. Hence, (7.9) gives $p_{nj+t} = 2$. This completes the induction step. Now we divide the proof of this case into three subcases.

Subcase 2.1. $2r - 2 > n$.

By Lemma 7.2,

$$\begin{aligned} & p_{nj+n} + \sum_{h=0}^{j-1} p_{nh+n} - 2 \sum_{h=1}^j p_{nh+n-1} + \sum_{h=1}^j p_{nh+n-2} \\ & + \sum_{h=1}^{n-1} s_{nj+h} p_{n-h} + (nj + n) s_{nj+n} = 0. \end{aligned}$$

We have proved that $p_{nj+s} = 2$ for $r + 1 \leq n - 1$. Therefore,

$$p_{nj+n} + (2 - n)j - 4j + 2j - 2 \cdot 2 + 2 + (nj + n) s_{nj+n} = 0.$$

Observe that $2 < 2r - n < 2n - n = n$, so, by Lemma 7.2, $s_{nj+n} = s_{2g-(nj+n)} = s_{nj+2r-n} = 0$. Hence $p_{nj+n} = 2 + nj$. By (2.3), $L(f^{nj+n}) = -nj < 0$. Since $nj + n \leq g - 3 + g - 3 < 3g - 3$ the desired conclusion holds in this case.

Subcase 2.2. $2r - 2 = n$.

By Lemmas 7.2 and 7.2,

$$P(x) = (x - 1)^2(x^{n(2j+1)} + x^{2nj} + \dots + x^n + 1) = \frac{x^{2nj+2n} - 1}{x^n - 1}(x - 1)^2.$$

Observe that $2nj + 2n \leq 2g - 6 + g - 3 < 3g - 3$. Since, as above, we can see that the $2(nj + n)$ -th powers of the eigenvalues of f_{*1} are 1, from Lemma 7.5 we can deduce that $L(f^{2nj+2n}) < 0$. Taking $m = 2nj + 2n$, we can complete the proof.

Subcase 2.3. $2r - 2 < n$.

By Lemma 7.2 (taking $v = g + 1$),

$$\begin{aligned} p_{nj+2r-2} + \sum_{h=0}^{j-1} p_{nh+2r-2} - 2 \sum_{h=1}^j p_{nh+2r-3} + \sum_{h=1}^j p_{nh+2r-4} \\ + \sum_{h=1}^{2r-3} s_{nj+h} p_{2r-2-h} + (nj + 2r - 2) s_{nj+2r-2} = 0. \end{aligned}$$

Since $2 \leq 2r - 4 < 2r - 3 < 2r - 2 < n$, $p_{nh+2r-3} = p_{nh+2r-4} = 2$ if $0 \leq h \leq j$ and $p_{nh+2r-2} = 2$ if $0 \leq h < j$. Further, $s_{nj+1} = -2$ and $s_{nj+2} = 1$. Hence,

$$p_{nj+2r-2} + 2j - 4j + 2j - 4 + 2 + (nj + 2r - 2)1 = 0.$$

So $p_{nj+2r-2} = 2 - (nj + 2r - 2)$ and $L(f^{nj+2r-2}) = nj + 2r - 2 \neq 0$. This assures the existence of a periodic orbit whose period divides $l = nj + 2r - 2$. Since $n + 4 \leq nj + 2r - 2 < nj + n$, n does not divide $nj + 2r - 2$, so the period of the orbit is $nj + 2r - 2$. Now observe that $r \geq 3$, so $nj + 2r - 2 \geq g + 1$. Moreover, there are no periodic orbits of period $g + 1$. Therefore, $nj + 2r - 2 \geq g + 2$. Since $j \geq 1$,

$$3(nj + 2r - 2) = 3nj + 3r + 2r - 2 + r - 4 < 3g + g - r + r - 4 = 4g - 4.$$

Thus, taking l as $nj + 2r - 2$ the desired conclusion holds for this subcase. ■

7.4.2 The orientation-reversing case

The following four lemmas show results analogous to Lemmas 7.1, 7.2 and Proposition 7.3 for the orientation-reversing case. Since, by Lemma 7.2, the equation relating the values of the s_i 's corresponding to an orientation-reversing map of Σ_g depends on the parity of g , the analogue of Proposition 7.3 for the orientation-reversing case, splits into two results, Propositions 7.6 and 7.7. The proof of the following lemma is analogous of the proof of Lemma 7.1.

Lemma 7.4 *Let $A \in GL_k(\mathbb{Z})$ be such that, for some positive integer n and some non-negative integer j , the coefficients s_i of the characteristic polynomial of A satisfy,*

$$s_i = \begin{cases} 1 & \text{if } i \equiv_n 0, \\ -1 & \text{if } i \equiv_n 2, \\ 0 & \text{otherwise,} \end{cases}$$

for $i \leq nj$. Then, for each positive integer t such that $nj + t \leq k$,

$$p_{nj+t} + \sum_{h=0}^{j-1} p_{nh+t} - \sum_{h=1}^j p_{nh+t-2} + \sum_{h=1}^{t-1} s_{nj+h} p_{t-h} + (nj+t)s_{nj+t} = 0.$$

Lemma 7.5 *Let $g \geq 2$ and let $f \in \mathcal{H}_g^-$. If there exist positive integers n and v such that n is even, $n \geq 3$ and $v \leq 2g$ such that, for every $i \leq v$,*

$$L(f^i) = \begin{cases} n & \text{if } n \text{ divides } i, \\ 0 & \text{otherwise,} \end{cases}$$

then, for $i \leq v$,

$$s_i = \begin{cases} -1 & \text{if } i \equiv_n 2, \\ 1 & \text{if } i \equiv_n 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, if j, t are positive integers such that $nj \leq v$ and $nj + t \leq 2g$ then

$$p_{nj+t} + \sum_{h=0}^{j-1} p_{nh+t} - \sum_{h=1}^j p_{nh+t-2} + \sum_{h=1}^{t-1} s_{nj+h} p_{t-h} + (nj+t)s_{nj+t} = 0.$$

Proof. Assume that we have proved the first statement of the lemma. Then, the second one follows by Lemma 7.4. Now, let us prove the formula for the s_i 's.

By (2.3),

$$(7.10) \quad p_i = \begin{cases} 2 - n & \text{if } n \text{ divides } i, \\ 0 & \text{if } i \text{ is odd,} \\ 2 & \text{otherwise,} \end{cases}$$

for $i \leq v$.

Now we proceed by induction. By Newton's equation (N.1) (page 22), (7.10) holds for $i = 1$. Consider now $1 \leq i \leq v - 1$ and suppose that the statement holds for s_1, s_2, \dots, s_i . Let j and r be the non-negative integers such that $i = nj + r$ and $0 \leq r < n$. Recall that a matrix representing f_{*1} has size $2g \times 2g$ and observe that $2 \leq i + 1 = nj + r + 1 \leq v \leq 2g$. By the induction hypothesis and Lemma 7.4,

$$(7.11) \quad p_{i+1} + \sum_{h=0}^{j-1} p_{nh+r+1} - \sum_{h=1}^j p_{nh+r-1} + \sum_{h=1}^r s_{nj+h} p_{t-h} + (i+1)s_{i+1} = 0.$$

Since n is even, if r is even, then $nj + r - 1$ and $nj + r + 1$ are odd for every positive integer j . Therefore, all the terms of the above sum except $(i+1)s_{i+1}$ are equal to 0. So, $s_{i+1} = 0$. So we can assume that r is odd. Considering separately the three cases, $r = 1, n - 1$ and $r \notin \{1, n - 1\}$, we show that the values of the p_l 's for $l \in \{1, 2, \dots, i + 1\}$ and s_l 's for $l \in \{nj + 1, nj + 2, \dots, nj + r\}$ can be replaced to find s_{i+1} . Indeed, if $r = 1$, then (7.11) can be written as

$$p_{i+1} + 2j - j(2 - n) + (nj + 2)s_{i+1} = 0,$$

which gives $s_{i+1} = -1$.

If $r = n - 1$ then (7.11) gives

$$p_{i+1} + j(2 - n) - 2j + 2 + (nj + 1)s_{i+1} = 0,$$

so, $s_{i+1} = 1$. Finally, if r is odd and different from 1 and $n - 1$, by (7.11),

$$2 + 2j - 2j - 2 + (i + 1)s_{i+1} = 0,$$

which gives $s_{i+1} = 0$. ■

The following proposition will be used in the proof of Proposition 11.3.

Proposition 7.6 *Let g be a positive even integer such that $g \geq 6$, and let $f \in \mathcal{H}_g^-$ satisfying the following conditions:*

- (1) $\text{Per}(f) \cap \{1, 2, 3, 4, 5, g+1, g+2\} = \emptyset$.
- (2) *There exists an even positive integer n such that $\text{Per}(f) \cap \{6, 7, \dots, g\} = \{n\}$. Moreover, there is only one orbit of period n and each of its points has index 1 for f^{hn} for each positive integer h such that $hn \leq g$.*

Then there exists a positive integer m such that $m \leq 2g - 6$ and $L(f^m) < 0$.

Proof. By the same arguments as in the proof of Proposition 7.3 we find

$$L(f^i) = \begin{cases} n & \text{if } n \text{ divides } i, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, by (2.3),

$$(7.12) \quad p_i = \begin{cases} 2 - n & \text{if } n \text{ divides } i, \\ 0 & \text{if } i \text{ is odd,} \\ 2 & \text{otherwise,} \end{cases}$$

for $i \leq g + 2$.

Let r, j be the non-negative integers such that $g = nj + r$ and $0 \leq r \leq n - 1$. Since $n \leq g$, $j \geq 1$. Observe that since n and g are even so is r . In particular, $r \notin \{1, 3\}$. We split the proof into three cases.

Case 1. $r = 0$.

Here, $g = nj$. Since $g + 2 \leq 2g$, by Lemma 7.5,

$$p_{g+2} + \sum_{h=0}^{j-1} p_{nh+2} - \sum_{h=1}^j p_{nh} + \sum_{h=1}^1 s_{nj+h} p_{2-h} + (g+2)s_{g+2} = 0.$$

By (7.12), $p_{g+2} + nj + (g+2)s_{g+2} = 0$. By Lemmas 7.2 and 7.5, $s_{g+2} = s_{2g-(g+2)} = s_{g-2} = s_{n(j-1)+n-2} = 0$ because $2 < n-2 < n$. Therefore, $p_{g+2} = -nj = -g$ and, by (2.3), $L(f^{g+2}) = g+2$. This implies the existence of an orbit whose period divides $g+2 = nj+2$. But this contradicts our hypothesis because the only orbit of period at most $g+2$ is the orbit of period n and since $n > 2$, n does not divide $g+2$.

Case 2. $r = 2$.

By Lemma 7.5,

$$p_{g+2} + \sum_{h=0}^{j-1} p_{nh+4} - \sum_{h=1}^j p_{nh+2} + \sum_{h=1}^3 s_{nj+h} p_{4-h} + (g+2)s_{g+2} = 0.$$

Since $n > 4$ is even, $p_{nh+4} = 2$ for $h \in \{0, 1, \dots, j-1\}$ and $p_{nh+2} = 2$ for $h \in \{0, 1, \dots, j\}$. By Lemma 7.5, $s_g = s_{mj+2} = -1$. Then

$$p_{g+2} + 2j - 2j - 2 + (g+2)s_{g+2} = 0.$$

By Lemmas 7.2 and 7.5, $s_{g+2} = s_{2g-(g+2)} = s_{g-2} = s_{nj} = 1$. So, $p_{g+2} = -g$ and $L(f^{g+2}) = g+2$. Since $g+2 = nj+4$ and $n \geq 6$, by an argument analogous to that of the previous case, we obtain a contradiction.

Case 3. $r > 2$.

We will show by induction on t that if $r+1 \leq t < \min\{n, 2r-2\}$ then

$$(7.13) \quad p_{nj+t} = \begin{cases} 2 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

Since $nj+r+1 = g+1 < g+2$ and $4 \leq r+1 \leq t < n$, n does not divide $g+1$, by (7.12), the first step of the induction. Now fix $r+1 < t < \min\{n, 2r-2\}$ and assume that (7.13) holds for every $r+1 \leq s < t$. Observe that $nj+t < nj+n < 2g$. Then, by Lemma 7.5, (taking $v = g$),

$$p_{nj+t} + \sum_{h=0}^{j-1} p_{nh+t} - \sum_{h=1}^j p_{nh+t-2} + \sum_{h=1}^{t-1} s_{nj+h} p_{t-h} + (nj+t)s_{nj+t} = 0.$$

Since n is even, if t is odd all the terms of the form $s_{nj+h} p_{t-h}$ for $h \in \{1, 2, \dots, t-1\}$ are equal to 0, because $s_{nj+h} = 0$ for each $h \in \{1, 3, \dots, t-1\}$ and $p_{t-2} = 0$. By Lemma 7.2, $s_{nj+t} = s_{2g-(nj+t)} = s_{g+2r-t} = 0$ because $2 < 2r-t < 2r-(r+1) < n$. Thus $p_{nj+t} = 0$. Assume now that t is even. Since $4 \leq r+1 < t < n$ we have $t-2, t \not\equiv_n 0$. By Lemmas 7.5 and 7.2 $s_{nj+2} = -1$ and $s_{nj+l} = 0$ if $l \in \{1, 3, 4, \dots, t-1\}$. Then, substituting these values in the above equation, and using (7.12) and the induction hypothesis, we obtain

$$p_{nj+t} + 2j - 2j - 2 + s_{nj+t}(nj+t) = 0.$$

Since $2 < 2r-t < n$, by Lemma 7.2, $s_{nj+t} = s_{2g-(nj+t)} = s_{nj+2r-t} = 0$. So, $p_{nj+t} = 2$ as desired. Now we split the proof of this case into two subcases.

Subcase 3.1. $2r - 2 \geq n$.

By Lemma 7.5,

$$p_{nj+n} + \sum_{h=0}^{j-1} p_{nh+n} - \sum_{h=1}^j p_{nh+n-2} + \sum_{h=1}^{n-1} s_{nj+h} p_{n-h} + (nj+n)s_{nj+n} = 0.$$

Since $n \geq 3$, $p_{nh+n-2} = 2$ and $p_{nh+n} = 2 - n$ for $0 \leq h < j$. Further, we have proved that $p_{nj+n-2} = 2$. So, by (7.12),

$$p_{nj+n} + (2-n)j - 2j - 2 + (nj+n)s_{nj+n} = 0.$$

Since $2 \leq 2r - n < n$, by Lemma 7.2, $s_{nj+n} = s_{2g-(nj+n)} = s_{nj+2r-n} \in \{0, -1\}$. So, $p_{nj+n} \in \{2 + nj, 2 + 2nj + n\}$. In all cases, $L(f^{nj+n}) < 0$ which completes this subcase because $nj + n < 2g - 2r \leq 2g - 6$.

Subcase 3.2. $2r - 2 < n$.

By Lemma 7.5,

$$\begin{aligned} p_{nj+2r-2} + \sum_{h=0}^{j-1} p_{nh+2r-2} - \sum_{h=1}^j p_{nh+2r-4} \\ + \sum_{h=1}^{2r-3} s_{nj+h} p_{2r-2-h} + (nj+2r-2)s_{nj+2r-2} = 0. \end{aligned}$$

By Lemmas 7.5 and 7.2, $s_{nj+l} = 0$ if $l \in \{1, 3, \dots, 2r-3\}$. Since $r \geq 3$, $2 \leq 2r-4 < 2r-2 < n$. Thus, $p_{nh+2r-2} = p_{nh+2r-4} = 2$ if $h \in \{0, 1, \dots, j-1\}$. Also, $p_{nj+2r-4} = 2$. Further, $s_{nj+2} = -1$. Hence

$$p_{nj+2r-2} + 2j - 2j - 2 + (nj+2r-2)s_{nj+2r-2} = 0.$$

By Lemmas 7.2 and 7.5, $s_{nj+2r-2} = s_{2g-(nj+2r-2)} = s_{nj+2} = -1$. So, $p_{nj+2r-2} = 2 + nj + 2r - 2$. By (2.3), $L(f^{nj+2r-2}) = -(nj+2r-2) < 0$. Since $nj + 2r - 2 < g - r + n \leq g - r + g - r \leq 2g - 6$, we are done. \blacksquare

The following result will be used in the proof of Proposition 11.2

Proposition 7.7 *Let g be a positive odd integer such that $g \geq 3$ and let $f \in \mathcal{H}_g^-$ satisfy the following conditions:*

(1) $1, 2 \notin \text{Per}(f)$.

(2) *There exists an even positive integer n such that $\text{Per}(f) \cap \{3, 4, \dots, g\} = \{n\}$. Moreover, there is only one orbit of period n and each of its points has index 1 for f^{hn} for each positive integer h such that $hn \leq g$.*

Then either there exists m such that $1 \leq m \leq 3g - 3$ and $L(f^m) < 0$, or there exists $l \in \text{Per}(f)$ such that $g + 1 \leq l \leq \frac{4g-4}{3}$.

Proof. Using the same arguments as in the proof of Proposition 7.3 we find

$$L(f^i) = \begin{cases} n & \text{if } n \text{ divides } i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i \leq g$. Consequently, by (2.3),

$$(7.14) \quad p_i = \begin{cases} 2 - n & \text{if } n \text{ divides } i, \\ 0 & \text{if } i \text{ is odd,} \\ 2 & \text{otherwise,} \end{cases}$$

for $i \leq g$.

Let r, j be the non-negative integers such that $g = nj + r$ and $0 \leq r \leq n - 1$. Since $n \leq g$, $j \geq 1$. Observe that n is even and g is odd, so r must be odd. In particular, $r \notin \{0, 2\}$. We split the proof into three cases.

Case 1. $r = 1$.

By Lemmas 7.5 and 7.2

$$P(x) = (x^2 - 1)(x^{2nj} + x^{n(2j-1)} + \dots + x^n + 1) = \frac{x^{2nj+n} - 1}{x^n - 1}(x - 1)^2.$$

As in the proof of Proposition 7.3 we can see that this implies that the $(2nj + n)$ -th power of each of the eigenvalues of f_{*1} is 1 and since $2nj + n \leq 2g - 2 + g - 1 = 3g - 3$, we are done.

Case 2. $r \geq 3$.

By an argument analogous to that of the Case $r \geq 3$ of the proof of Proposition 7.6 (doing the first step of the induction with r instead of $r + 1$) we can show by induction on t that if $r \leq t < \min\{n, 2r - 2\}$ then

$$(7.15) \quad p_{nj+t} = \begin{cases} 2 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

Now we split the proof of this case into three subcases.

Subcase 2.1. $2r - 2 = n$.

By Lemmas 7.5 and 7.2,

$$P(x) = (x^2 - 1)(x^{2nj+n} + x^{n(2j-1)} + \dots + x^n + 1) = \frac{x^{2nj+2n} - 1}{x^n - 1}(x - 1)^2.$$

As in the proof of Proposition 7.3, we can see that this implies that the $(2nj + 2n)$ -th power of each eigenvalue of f_{*1} is 1. Since $2nj + 2n = 2g - 2r + n + n \leq 2g - 2 + g - 3 = 3g - 5 < 3g - 3$, we are done.

Subcase 2.2. $2r - 2 > n$.

Using the same arguments as in the Case $2r - 2 \geq n$ of the proof of Proposition 7.6 we can prove that $p_{nj+n} + (2 - n)j - 2j - 2 + (nj + n)s_{nj+n} = 0$. It follows from Lemma 7.2 that $s_{nj+n} = -s_{2g-(nj+n)} = -s_{nj+2r-n} = 0$ because $2 < 2r - n < n$. Thus $p_{nj+n} = 2 + nj$, so $L(f^{nj+n}) = -nj < 0$. Since $nj + n \leq 2g - 6 < 3g - 3$, we are done.

Subcase 2.3. $2r - 2 < n$.

By an argument analogous to that for the Case $2r - 2 < n$ of the proof of Lemma 7.3, we obtain

$$p_{nj+2r-2} + 2j - 2j - 2 + (nj + 2r - 2)s_{nj+2r-2} = 0.$$

By Lemmas 7.2 and 7.5, $s_{nj+2r-2} = -s_{2g-(nj+2r-2)} = -s_{nj+2} = 1$, so $p_{nj+2r-2} = 2 - (nj + 2r - 2)$. By (2.3), $L(f^{nj+2r-2}) = nj + 2r - 2$. As in Case $2r - 2 < n$ of the proof Proposition 7.3, this assures the existence of a periodic orbit of period $l = nj + 2r - 2 \geq g + 1$ such that $3l \leq 4g - 4$. ■

Chapter 8

Consequences of the Thurston-Nielsen theory

This chapter is devoted to studying some results about homeomorphisms in Thurston canonical form and in standard form. In Section 8.1 we state some properties of pseudo-Anosov maps. In Section 8.2 we determine sufficient conditions for the existence of fixed-point classes of negative index of iterates of pseudo-Anosov maps and reducible maps in standard form, and, in Section 8.3, we study finite-order maps and reducible maps.

8.1 Properties of pseudo-Anosov maps

The following lemmas are consequences of the Euler-Poincaré Formula (3.2), and they will be used in Chapter 11.

Lemma 8.1 *If there exists a pseudo-Anosov map $f: \Sigma_{0,b} \longrightarrow \Sigma_{0,b}$ then $b \geq 4$.*

Proof. Suppose f exists. Let \tilde{b} be the number of one-pronged boundary components of the foliation on $\Sigma_{0,b}$. Let B' be a boundary component from which emanates more than one prong and $p_{B'}$ denote the number of such prongs. Then by Remark 3.2, $\sum_{s \in \text{Sing}(B')} = -p_{B'} \leq -2$. Since there are $b - \tilde{b}$ such components, by the Euler-Poincaré Formula (3.2),

$$2(2 - b) = 2\chi(\Sigma_{0,b}) = \sum_{s \in \text{Sing}(\Sigma)} (2 - p_s)$$

$$\leq \sum_{s \in \text{Sing}(\partial(\Sigma))} (2 - p_s) \leq -\tilde{b} - 2(b - \tilde{b}) = \tilde{b} - 2b.$$

Thus $4 \leq \tilde{b}$ and, since $\tilde{b} \leq b$, the result follows. \blacksquare

Lemma 8.2 *Let $g \neq 1$ and let $f: \Sigma_g \rightarrow \Sigma_g$ be a pseudo-Anosov map. If k is the number of f -orbits of singularities of the foliation on Σ_g then $k \geq 1$.*

Proof. If $k = 0$ then the foliation has no singularities. Hence, by the Euler-Poincaré Formula (3.2), $g = 1$, which is impossible. \blacksquare

8.2 Fixed-point classes of iterates of maps in standard form

This section is devoted to stating some conditions which ensure the existence of fixed-point classes of negative index for iterates of pseudo-Anosov maps and reducible maps in standard form. Most of these results are consequences of the characterization of fixed-point classes given in Chapter 4, mainly in Proposition 4.4.

Lemma 8.1 *Let p and n be positive integers and let $f: \Sigma_g \rightarrow \Sigma_g$ be a pseudo-Anosov map in standard form. If x is a p -pronged periodic point of period n then the following holds.*

- (1) *If f preserves orientation then $\text{Ind}_{f^{pn}}(x) = 1 - p$.*
- (2) *If f reverses orientation and n is odd then $\text{Ind}_{f^{2n}}(x) = 1 - p$.*
- (3) *If x is a regular point then $\text{Ind}_{f^{2n}}(x) = 1 - 2 = -1$.*

Proof. Observe that x is a fixed point of f^n . Moreover, f^n is orientation-reversing if and only if f is orientation-reversing and n is odd. Suppose first that f preserves orientation. By Proposition 4.1, x is of type $(p, 0)^+$ for f^{np} and, by Proposition 4.2, $\text{Ind}_{f^{np}}(x) = 1 - p$. On the other hand, if f reverses orientation and n is odd, by Proposition 4.1, x is of type $(p, 0)^+$ for f^{2n} . Hence, as before, $\text{Ind}_{f^{2n}}(x) = 1 - p$. Finally, the third statement follows from the first two. \blacksquare

Proposition 8.2 *Let $f: \Sigma \rightarrow \Sigma$ be an orientation-preserving homeomorphism in standard form. Assume that f has a pseudo-Anosov component with*

either an interior fixed point or an invariant boundary component. Moreover, suppose that every prong emanating from the fixed point or invariant boundary component is fixed under the action of f . Then f has a fixed-point class of negative index.

Proof. By Proposition 4.2, an interior fixed point x of a pseudo-Anosov component is a fixed-point class. Moreover, if p is the number of prongs emanating from x , then x is of type $(p, 0)^+$ because, by hypothesis, all the prongs emanating from x remain fixed under the action of f . By Proposition 4.2, the index of x is $1 - p$. Since $p \geq 2$, this index is negative.

Now suppose that f has an invariant boundary component B such that all prongs remain fixed under the action of f . By Remark 4.3, $f|_B$ is a rotation which implies that $f|_B = \text{Id}$. Therefore, B is contained in a fixed-point class C . This class C must be as in Proposition 4.4(C.2), (C.3) or (F). In each of the three cases, $\text{Ind}_f(C) \leq -p < 0$, where p is the number of prongs emanating from B . ■

Lemma 8.3 *Let m and g be positive integers such that $g \geq 2$, and let $f: \Sigma_g \rightarrow \Sigma_g$ be a pseudo-Anosov map in standard form.*

- (1) *If there exists $x \in \Sigma_g$ such that x is an isolated fixed point of f^m , m is even if f is orientation-reversing, and $\text{Ind}_{f^m}(x) \neq 1$, then f^m has a fixed-point class of negative index.*
- (2) *If $L(f^m) < 0$ then f^m has a fixed-point class of negative index.*
- (3) *If there exists a singularity x such that $f^m(x) = x$ and each prong emanating from x remains fixed under the action of f^m , then f^m has a fixed-point class of negative index.*
- (4) *If there exists a p -pronged singularity x of period n , n is even if f reverses orientation, and $m = np$, then f^m has a fixed-point class of negative index.*
- (5) *If m is even, $m/2$ is odd, f is orientation-reversing and there exists a singularity of period $m/2$, then f^m has a fixed-point class of negative index.*
- (6) *If m is even and there exists a regular point of period $m/2$ then f^m has a fixed-point class of negative index.*

Proof. Observe that by Proposition 4.2, isolated fixed points of pseudo-Anosov maps on closed surfaces are fixed-point classes. Therefore, it suffices to show that, in each case, f^m has a fixed point of negative index.

Suppose that the hypotheses of (1) hold. Let us denote by p the number of prongs emanating from x . By Proposition 4.2, if m is even or f is orientation-preserving, $\text{Ind}_{f^m}(x) \in \{1, 1 - p\}$. Since $p \geq 2$ and $\text{Ind}_{f^m}(x) \neq 1$ we have that $\text{Ind}_{f^m}(x) = 1 - p \leq -1$.

If the hypothesis of (2) holds, then the conclusion follows from Theorem 2.9.

The statement (3) is a consequence of Proposition 8.2 and the statements (4), (5) and (6) are consequences of Lemma 8.1. \blacksquare

Lemma 8.4 *Let $g \geq 2$, let $f \in \mathcal{H}_g$ be a pseudo-Anosov map in standard form, and let k be the number of f -orbits of singularities of the foliation on Σ_g .*

- (1) *If $k \geq 3$ then there exists a positive integer m such that $m \leq 4g - 4$ and f^m has a fixed-point class of negative index.*
- (2) *If $k = 2$, $n_1 \leq n_2$ and $p_2 \geq 4$ where n_1, n_2 are the periods of the orbits of singularities and p_2 is the number of prongs emanating from each point in the orbit of period n_2 then there exists a positive integer m such that $m \leq 4g - 4$ and f^m has a fixed-point class of negative index.*

Proof. Let us denote by O_1, O_2, \dots, O_k be the f -orbits of singularities. For each $i \in \{1, 2, \dots, k\}$, let n_i be the period of O_i , and let p_i be the number of prongs emanating from each element of O_i . Hence, we may write the Euler-Poincaré Formula (3.2) in the form,

$$\sum_{i=1}^k n_i(p_i - 2) = 4(g - 1).$$

Assume that $k \geq 3$. Since $p_i \geq 3$, for each i ,

$$(8.1) \quad \sum_{i=1}^k n_i \leq \sum_{i=1}^k n_i(p_i - 2) = 4(g - 1).$$

Let $i_0 \in \{1, 2, \dots, k\}$ be such that $n_{i_0}p_{i_0} = \min_{1 \leq i \leq k} \{n_i p_i\}$. Then,

$$3n_{i_0}p_{i_0} \leq \sum_{i=1}^k n_i p_i = 4(g - 1) + 2 \sum_{i=1}^k n_i \leq 12(g - 1),$$

i.e., $n_{i_0}p_{i_0} \leq 4(g-1)$. If n_{i_0} is even or f is orientation-preserving then Lemma 8.3(4) holds for $m = n_{i_0}p_{i_0}$. If n_{i_0} is odd and f is orientation-reversing then Lemma 8.3(5) holds for $m = 2n_{i_0}$. Hence, (1) is proved.

Let us prove (2). By (8.1),

$$n_1p_1 \leq n_1(p_1 - 2) + 2n_2 \leq n_1(p_1 - 2) + n_2(p_2 - 2) = 4(g-1).$$

As in the preceding paragraph, this gives a proof of (2). ■

Lemma 8.5 *Let m, n, p, g be positive integers such that $n \geq 2$ and $g \geq 2$, and let $f \in \mathcal{H}_g$ be a reducible map in standard form.*

- (1) *Suppose there exist a pseudo-Anosov f -component C and a p -pronged boundary component B of C such that, $f^n(B) = B$, $pn = m$, and n is even or f is orientation-preserving. Then f^m has a fixed-point class of negative index.*
- (2) *Suppose there exist a pseudo-Anosov f -component C , and a boundary component B of C such that $f^n(B) = B$, n is odd, $2n = m$, and f is orientation-reversing. Then f^m has a fixed-point class of negative index.*
- (3) *Suppose there exist a pseudo-Anosov f -component C , and $x \in \text{Int}(C)$, such that $f^n(x) = x$, n is odd, $2n = m$, and f is orientation-reversing. Then f^m has a fixed-point class of negative index.*
- (4) *Suppose there exist a pseudo-Anosov f -component C , and a p -pronged boundary singularity $x \in C$ such that, $f^n(x) = x$, $pn = m$, and n is even or f is orientation-preserving. Then f^m has a fixed-point class of negative index.*
- (5) *Suppose there exists a finite-order f -component C such that $f^m|_C = \text{Id}_C$. Then f^m has a fixed-point class of negative index.*

Proof. The statements (1), (2), (3) and (4) are consequences of Lemma 8.2 and Proposition 4.1.

If there exists a finite-order component C such that $f^m|_C = \text{Id}_C$ then C is included in a fixed-point class \tilde{C} of f^m . Recall that by definition, a component of a reducible map has negative Euler characteristic. Then, by Proposition 4.4(F), $\text{Ind}_{f^m}(\tilde{C}) \leq \chi(C) < 0$. Hence, (5) is proved. ■

Remark 8.6 Despite the title of this section, these results (and their consequences) hold for maps in Thurston canonical form. Indeed, observe that

if we have a fixed point, or an invariant boundary component, of a pseudo-Anosov map in Thurston canonical form $f: \Sigma \longrightarrow \Sigma$, we can define type in the same way as we did for maps in standard form: A p -pronged fixed point or a p -pronged invariant boundary component will be of type $(p, k)^+$ (resp. $(p, k)^-$) if f preserves (resp. reverses) orientation and f acts as the map $r_{(p,k)}^+$ (resp. $r_{(p,k)}^-$) on the prongs emanating from it. By Lemma 4.1, there exists a pseudo-Anosov homeomorphism $g: \Sigma \longrightarrow \Sigma$ in standard form isotopic to f . The isotopy between these two maps preserves the types of fixed points and invariant boundary components. Since, by Lemma 2.8, indices of fixed-point classes are preserved under isotopy, the index of a fixed point, or of an invariant boundary component, of any pseudo-Anosov map can be calculated as a function of its type according to Tables 4.1 and 4.2. However, the more restrictive statements for homeomorphisms in standard form are sufficient for our purposes. \square

8.3 Reducible maps

This section is devoted to the study of reducible maps. In Subsection 8.3.1 we state some properties of a system of reducing curves. In Subsection 8.3.2 (resp. 8.3.3) we determine some properties of finite-order components of orientation-preserving (resp. orientation-reversing) reducible maps.

8.3.1 Components of reducible maps

We begin with a basic property.

Remark 8.1 If A and B are surfaces then

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B),$$

see [10, Corollary V.4.6 and Proposition V.5.8]. \square

By using the preceding remark, we shall prove the next result, which belongs to the class of well-known facts whose proof is hard to find in the literature. Before stating it, let us introduce the following notation: For each Σ , denote by $\text{genus}(\Sigma)$ (resp. $\text{bc}(\Sigma)$) its genus (resp. its number of boundary components).

Lemma 8.2 *Let $\Gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_n\}$ be a finite set of pairwise disjoint simple closed curves in a surface Σ and let $\Sigma^1, \Sigma^2, \dots, \Sigma^k$ denote the closure of the connected components of $\Sigma \setminus \Gamma$. Then $\chi(\Sigma) = \sum_{i=1}^k \chi(\Sigma^i)$. Further,*

$$\sum_{i=1}^k \text{genus}(\Sigma^i) \leq \text{genus}(\Sigma).$$

Proof. We will prove by induction that if $j \leq k$, then

$$\chi(\cup_{i=1}^j \Sigma^i) = \sum_{i=1}^j \chi(\Sigma^i).$$

That equality holds when $j = 1$ is trivial. We now assume that the formula holds for some positive integer j such that $j < k$. The intersection of $\cup_{i=1}^j \Sigma^i$ with Σ^{j+1} is a finite (possibly empty) union of pairwise disjoint simple closed curves. In any case, the Euler characteristic of such an intersection is 0, so by Remark 8.1 and the inductive hypothesis,

$$\begin{aligned} \chi\left(\bigcup_{i=1}^{j+1} \Sigma^i\right) &= \chi\left(\bigcup_{i=1}^j \Sigma^i\right) + \chi(\Sigma^{j+1}) - \chi\left(\left(\bigcup_{i=1}^j \Sigma^i\right) \cap \Sigma^{j+1}\right) \\ &= \sum_{i=1}^j \chi(\Sigma^i) + \chi(\Sigma^{j+1}) = \sum_{i=1}^{j+1} \chi(\Sigma^i) \end{aligned}$$

as desired.

We prove now that $\sum_{i=1}^k \text{genus}(\Sigma^i) \leq \text{genus}(\Sigma)$. The surface $\Sigma = \cup_{i=1}^k \Sigma^i$ is connected, therefore, without loss of generality, we can assume that for each $i \in \{1, 2, \dots, k-1\}$,

$$\Sigma^i \cap \Sigma^{i+1} \neq \emptyset.$$

Since $\text{genus}(\cup_{i=1}^k \Sigma^i) = \text{genus}(\Sigma)$, it suffices to show that

$$(8.2) \quad \sum_{i=1}^j \text{genus}(\Sigma^i) \leq \text{genus}\left(\bigcup_{i=1}^j \Sigma^i\right)$$

for each $j \in \{1, 2, \dots, k\}$.

Clearly, (8.2), holds for $j = 1$. Let j be a positive integer such that $j \leq k$ and assume (8.2) holds for each $i \in \{1, 2, \dots, j-1\}$. Denote by

C_j the number of connected components of $(\cup_{i=1}^j \Sigma^i) \cap \Sigma^{j+1}$. Observe that our assumption implies that $C_j \geq 1$ for each $j \in \{1, 2, \dots, k-1\}$. By Remark 8.1, since $\chi((\cup_{i=1}^j \Sigma^i) \cap \Sigma^{j+1}) = 0$,

$$\chi\left(\bigcup_{i=1}^j \Sigma^i\right) + \chi(\Sigma^{j+1}) = \chi\left(\bigcup_{i=1}^{j+1} \Sigma^i\right).$$

Hence,

$$\begin{aligned} (8.3) \quad & 2\text{genus}\left(\bigcup_{i=1}^j \Sigma^i\right) + \text{bc}\left(\bigcup_{i=1}^j \Sigma^i\right) - 2 + 2g_{j+1} + b_{j+1} - 2 \\ &= 2\text{genus}\left(\bigcup_{i=1}^{j+1} \Sigma^i\right) + \text{bc}\left(\bigcup_{i=1}^{j+1} \Sigma^i\right) - 2. \end{aligned}$$

Each boundary component of $\cup_{i=1}^j \Sigma^{j+1}$ is either a boundary component of $\cup_{i=1}^j \Sigma^i$ or a boundary component of Σ^{j+1} . Further, $\cup_{i=1}^{j+1} \Sigma^i$ has exactly $\text{bc}(\cup_{i=1}^j \Sigma^i) - C_j$ boundary components lying in $\cup_{i=1}^j \Sigma^i$ and exactly $b_{j+1} - C_j$ boundary components lying in Σ^{j+1} . Thus, $\text{bc}(\cup_{i=1}^{j+1} \Sigma^i) = \text{bc}(\cup_{i=1}^j \Sigma^i) - C_j + b_{j+1} - C_j = \text{bc}(\cup_{i=1}^j \Sigma^i) + b_{j+1} - 2C_j$. Now, substituting this equality in (8.3) we obtain

$$2\text{genus}\left(\bigcup_{i=1}^j \Sigma^i\right) + 2g_{j+1} + 2C_j - 2 = 2\text{genus}\left(\bigcup_{i=1}^{j+1} \Sigma^i\right),$$

so

$$\text{genus}\left(\bigcup_{i=1}^j \Sigma^i\right) + g_{j+1} + C_j - 1 = \text{genus}\left(\bigcup_{i=1}^{j+1} \Sigma^i\right).$$

Since $C_j \geq 1$ the result follows from the inductive hypothesis. ■

We introduce the following notation which will be used frequently.

Notation 8.3 Let $g \geq 2$, let $f: \Sigma_g \rightarrow \Sigma_g$ be a reducible homeomorphism, let Γ be a system of invariant curves, and let $N(\Gamma)$ be an invariant tubular neighborhood for Γ . An f -transversal is a subset $\{C_1, C_2, \dots, C_k\}$ of f -components such that the set of all f -components is the disjoint union of the f -orbits of the C_i 's. For each $1 \leq i \leq k$, we denote by g_i , b_i , and n_i the genus, the number of boundary components, and the period of C_i , respectively. If $f^{n_i}|_{C_i}: C_i \rightarrow C_i$ is finite-order then, to simplify notation, we denote $\sigma_{f^{n_i}|_{C_i}}$ by σ_i . □

Lemma 8.4 *With Notation 8.3 the following hold.*

- (1) $\sum_{i=1}^k (2g_i + b_i - 2)n_i = 2g - 2$.
- (2) For each $i \in \{1, 2, \dots, k\}$, $2g_i + b_i - 2 \geq 1$.
- (3) For each $i \in \{1, 2, \dots, k\}$, $n_i \leq n_i(2g_i + b_i - 2) \leq 2g - 2$.
- (4) For each, $i \in \{1, 2, \dots, k\}$, if $g_i = 0$ then $b_i \geq 3$.

Proof. Observe that the set of all f -components is $\{f^j(C_i)\}_{1 \leq j \leq n_i, 1 \leq i \leq k}$ and that, since f is a homeomorphism, for each $i \in \{1, 2, \dots, k\}$ and each $j \in \{1, 2, \dots, n_i\}$, $\chi(C_i) = \chi(f^j(C_i))$. Thus (1) follows from Lemma 8.2.

By the definition of standard form, $2 - 2g_i - b_i = \chi(C_i) \leq -1$, hence (2) holds.

Clearly, (3) is a consequence of (1) and (2), and (4) a consequence of (2). ■

Now we study transversals with only one component.

Lemma 8.5 *With Notation 8.3, if there exists an f -transversal containing only one component, C_1 , and $n_1 \geq 2$, and the boundary components of C_1 form a cycle under the action of f^{n_1} , then $n_1 = 2$ and $g = 2g_1 + b_1 - 1$.*

Proof. Let B_1 be a boundary component of C_1 . By the hypothesis, the boundary components of C_1 are $\{f^{kn_1}(B_1)\}_{k \in \{0, 1, \dots, b_1-1\}}$.

Let $A \subset N(\Gamma)$ be the annulus which has B_1 as a boundary component and let C be an f -component such that $C \cap \text{Cl}(A) \neq B_1$ and $C \cap \text{Cl}(A) \neq \emptyset$. Since f is a homeomorphism, and $C = f^j(C_1)$ for some $j \in \{1, 2, \dots, n_1\}$, and the boundary components of C_1 form a cycle under the action of f^{n_1} , so do the boundary components of C . Therefore, each boundary component of C is of the form $f^{n_1 k}(\text{Cl}(A) \cap C)$ for some $k \in \{0, 1, \dots, b_1 - 1\}$.

Observe that $\bigcup_{k=0}^{b_1-1} f^{n_1 k}(A) \cup C \cup C_1$ is a closed subsurface of Σ_g . Therefore, $\bigcup_{k=0}^{b_1-1} f^{n_1 k}(A) \cup C \cup C_1 = \Sigma_g$. If $C = C_1$ then $n_1 = 1$. Since this is impossible, $n_1 = 2$, and, by Lemma 8.4(1), $g = 2g_1 + b_1 - 1$ and $n_1 = 2$, as desired. ■

If $f \in \mathcal{F}_{g,b}$ then $\sigma_f = \sigma_{\tilde{f}}$. Then, by Theorem 3.1 we have the following result.

Corollary 8.6 *If $g \geq 2$ and $f \in \mathcal{F}_{g,b}^+$ (resp. $\mathcal{F}_{g,b}^-$) then $\sigma_f \leq 4g + 2$ (resp. $4g + (-1)^g 4$).*

8.3.2 The orientation-preserving case

We begin with a basic property of finite-order maps of Σ_0 .

Lemma 8.7 *If $f \in \mathcal{F}_0^+$ is not the identity, then f has exactly two fixed points, x_1 and x_2 . Therefore, the f -period of every $x \in \Sigma_0 \setminus \{x_1, x_2\}$ is σ_f .*

Proof. By (2.2), for every $m \in \mathbb{N}$, $L(f^m) = 2 - \text{trace } f_{*1}^m$. Since $H_1(\Sigma_0)$ is trivial, $\text{trace } f_{*1}^m = 0$, so $L(f^m) = 2$. Now, the result follows by Theorem 2.7 and Lemma 3.3. \blacksquare

As a consequence we have the following.

Corollary 8.8 *Let $f \in \mathcal{F}_{0,b}^+$. If $\sigma_f \geq 2$, and $b \geq 3$, then there exists a positive integer j and $a \in \{0, 1, 2\}$ such that $b = j\sigma_f + a$. In particular $b \geq \sigma_f$. Moreover, if $\sigma_f = b = 3$ then the three boundary components of $\Sigma_{0,b}$ form a cycle under the action induced by f .*

Proof. Consider the induced map $\tilde{f}: \Sigma_0 \rightarrow \Sigma_0$. By Lemma 8.7, each \tilde{f} -orbit has 1 or σ_f elements and there are exactly two \tilde{f} -orbits of only one element. The set of collapsed boundary components is \tilde{f} -invariant, so it is a disjoint union of orbits of \tilde{f} . Thus, $b = j\sigma_f + a$, where $j \in \mathbb{N}$ and $a \in \{0, 1, 2\}$. Clearly, if $b = \sigma_f = 3$, then $a = 0$, $j = 1$ and the three collapsed boundary components form an \tilde{f} -periodic orbit. Therefore, the boundary components of $\Sigma_{0,b}$ also form a cycle under the action of f . \blacksquare

Lemma 8.9 *Let $f: \Sigma_{1,b} \rightarrow \Sigma_{1,b}$ be an orientation-preserving finite-order map such that $\sigma_f > 2b$. Then $\sigma_f \in \{3, 4, 6\}$ and $b \in \{1, 2\}$. Moreover, if $b = 2$, then the two boundary components of $\Sigma_{1,b}$ are interchanged under the action of f .*

Proof. Consider the induced map $\tilde{f}: \Sigma_1 \rightarrow \Sigma_1$. If \tilde{f} is of type $[n; 0; \{ \}]$ for some positive integer n , then $\text{Per}(\tilde{f}) = \{\sigma_{\tilde{f}}\}$. With arguments analogous to the used in the proof of Corollary 8.8, we see that $b = j\sigma_f$ for some positive integer j . In particular, $b \geq \sigma_f$. Since this contradicts our hypotheses, \tilde{f} cannot be of type $[n; 0; \{ \}]$ and the result follows from Lemma 6.3. \blacksquare

Lemma 8.10 *If $g \geq 1$ and $f \in \mathcal{F}_g$.*

- (1) If $\sigma_f > 4g$ then f has at most one fixed point. Consequently, if $h \in \mathcal{H}_{g,1}^+$ (resp. $h \in \mathcal{H}_{g,2}^+$) is such that $\tilde{h} \in \mathcal{F}_g^+$ and $\sigma_h > 4g$, then h does not have fixed points (resp. both boundary components of $\Sigma_{g,2}$ are interchanged under the action of h).
- (2) If $\sigma_f > 2$ and f has an isolated fixed point then f is orientation preserving. Consequently, if $h \in \mathcal{H}_{g,1}^-$ and $\tilde{h} \in \mathcal{F}_g^-$ then $\sigma_h = 2$.

Proof. The second statement is a consequence of Lemma 3.5, so let us prove the first. Assume first that $g = 1$. Since $\sigma_f > 4$, by Lemma 6.3 f is either of type $[6; 0; \{1, 2, 3\}]$ or $[n; 0; \{ \}]$ for some $n \geq 5$ and the result follows directly. Therefore, we can assume that $g \geq 2$. Let $[\sigma_f; 0; \{p_1, p_2, \dots, p_R\}]$ be the type of f . If f has at least two fixed points, we can assume that $R \geq 2$ and $p_{R-1} = p_R = 1$. Set $R_0 = R - 2$ and $T = -R_0 + \frac{2g + \sum_{i=1}^{R-2} p_i}{\sigma_f}$. By Theorem 5.12, T is a non-negative integer. On the other hand, for each $i \in \{1, 2, \dots, R - 2\}$, $p_i \leq \frac{\sigma_f}{2}$. Hence, since by hypothesis $\sigma_f > 4g$,

$$R_0 \leq \frac{2g + \sum_{i=1}^{R_0} p_i}{\sigma_f} \leq \frac{2g + R_0 \frac{\sigma_f}{2}}{\sigma_f} < \frac{1 + R_0}{2}.$$

Then $R_0 < 1$, so $R_0 = 0$ and $T = \frac{2g}{\sigma_f}$. Since $\sigma_f > 4g$, T is not integer, which is impossible. \blacksquare

Lemma 8.11 *With Notation 8.3, let $i \in \{1, 2, \dots, k\}$ be such that C_i is a finite-order component.*

- (1) If $\sigma_i \leq 4g_i$ then $n_i \sigma_i \leq 4g$.
- (2) If $\sigma_i \leq 2$ then $n_i \sigma_i \leq 4g - 4$.

Proof. If $\sigma_i \leq 4g_i$, by Lemma 8.2, $\sigma_i n_i \leq 4g_i n_i \leq 4g$. Therefore, (1) holds.

If $\sigma_i \leq 2$, (2) follows directly from Lemma 8.4(3). \blacksquare

Lemma 8.12 *With Notation 8.3, let $i \in \{1, 2, \dots, k\}$ be such that C_i is a finite-order component of f and $f^{n_i}|_{C_i}$ is orientation preserving.*

- (1) If $g_i = 0$, and $b_i \neq 3$ or $\sigma_i \neq 3$ or $n_i \leq \frac{4}{3}(g - 1)$ then $n_i \sigma_i \leq 4g - 4$.
- (2) If $g_i = 1$, and $b_i \geq 3$ or $\sigma_i \geq 7$ then $n_i \sigma_i \leq 4g - 4$.
- (3) If $g_i \geq 2$, and $b_i \geq 3$ then $n_i \sigma_i \leq 4g - 4$.

Proof. Suppose now the hypotheses of (1) hold. By Lemma 8.11(2) we can assume that $\sigma_i \geq 3$. By Lemma 8.8, $3 \leq \sigma_i \leq b_i$. If $b_i = 3$, then $\sigma_i = 3$, $n_i \leq \frac{4}{3}(g-1)$ and the result follows directly. If $b_i \geq 4$, by Lemma 8.4(3), $2n_i \leq (b_i - 2)n_i \leq 2g - 2$. Therefore, $n_i\sigma_i \leq n_i(b_i - 2) + 2n_i \leq 4g - 4$, which completes the proof of (1).

Now, we prove (2). If $\sigma_i \leq 2b_i$ then by Lemma 8.4(3), $n_i\sigma_i \leq 2n_ib_i \leq 4g - 4$ as desired. If $\sigma_i > 2b_i$, the result holds by Lemma 8.9.

Finally, we prove (3). By Corollary 8.6, $\sigma_i \leq 4g_i + 2$. If $b_i \geq 3$, by Lemma 8.4(3),

$$n_i\sigma_i \leq 2n_i(2g_i + 1) \leq 2n_i(2g_i + b_i - 2) \leq 4g - 4,$$

as desired. ■

8.3.3 The orientation-reversing case

The aim of the present subsection is to obtain analogous results to those of the previous section for finite-order components of orientation-reversing reducible maps.

Remark 8.13 If $f: \Sigma \longrightarrow \Sigma$ is a finite-order orientation-reversing map then its order is even. □

Lemma 8.14 *If $b \geq 3$ and $f \in \mathcal{F}_{0,b}^-$ then there exist integers j and a such that $j \geq 1$, $a \in \{0, 2\}$, and $b = j\sigma_f/2 + a$. In particular, $b \geq \sigma_f/2$. Moreover, if $b = \sigma_f/2$, then $\sigma_f/2$ is odd and the b boundary components of $\Sigma_{0,b}$ form a cycle under the action of f .*

Proof. Consider the induced map $\tilde{f}: \Sigma_0 \longrightarrow \Sigma_0$. Clearly, \tilde{f}^m preserves orientation if m is even, and reverses orientation if m is odd. Since $H_1(\Sigma_0)$ is trivial, by (2.2),

$$L(\tilde{f}^m) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 2 & \text{if } m \text{ is even.} \end{cases}$$

That the result holds if $\sigma_f = 2$ is trivial. Therefore, we can assume without loss of generality that $\sigma_f > 2$. If $\text{Fix}(\tilde{f}) \neq \emptyset$ then, by Lemma 3.5, \tilde{f} has a pointwise-fixed simple closed curve γ and, \tilde{f} acts as a reflection in a

neighborhood of γ . Then, $\tilde{f}^2 = \text{Id}$ and $\sigma_{\tilde{f}} = \sigma_f = 2$, a contradiction. Then, we can assume that $\text{Fix}(\tilde{f}) = \emptyset$.

Now, by Remark 8.13, $h = \tilde{f}^2$ is an orientation-preserving map such that $\sigma_h = \sigma_f/2$ and $L(h^m) = 2$ for every $m \in \mathbb{N}$. By Lemma 8.7 it follows that h has two fixed points which are the only points of h -period strictly less than $\sigma_f/2$. Since $\text{Fix}(\tilde{f}) = \emptyset$, we can conclude that \tilde{f} has a two periodic orbit, which is the only periodic orbit of even period strictly less than σ_f .

By Lemma 3.6, if there exists a simple closed curve $\gamma \subset \text{Fix}(f^i)$ then i is odd and $i = \sigma_f/2$. In any case, $\text{Per}(\tilde{f}) \subset \{2, \sigma_f/2, \sigma_f\}$ and there is exactly one orbit of period 2. Since the set of collapsed boundary components is \tilde{f} -invariant, and has cardinal b , $b = k\sigma_f/2 + a$ for some $a \in \{0, 2\}$ and some $k \in \mathbb{N}$. Since $b \geq 3$, $k \geq 1$.

Now, observe that if $b = \sigma_f/2$, since $b \geq 3$, the set of collapsed boundary components must consist of an \tilde{f} -periodic orbit of period $\sigma_f/2$. Moreover, by Lemma 3.6, $\sigma_f/2$ is odd. Clearly, in this case, the boundary components of $f: \Sigma_{0,b} \rightarrow \Sigma_{0,b}$ form an f -cycle. \blacksquare

Lemma 8.15 *If $f \in \mathcal{F}_{1,b}^-$ then there exist a positive integer j such that $b = j\sigma_f/2$. In particular, $b \geq \sigma_f/2$.*

Proof. Consider the induced map $\tilde{f}: \Sigma_1 \rightarrow \Sigma_1$. The characteristic polynomial of \tilde{f}_{*1} is $x^2 - \text{trace}(\tilde{f}_{*1})x - 1$, whose roots are

$$\lambda_1 = \frac{\text{trace}(\tilde{f}_{*1}) + \sqrt{\text{trace}(\tilde{f}_{*1})^2 + 4}}{2}$$

and

$$\lambda_2 = \frac{\text{trace}(\tilde{f}_{*1}) - \sqrt{\text{trace}(\tilde{f}_{*1})^2 + 4}}{2}.$$

Since \tilde{f} is finite-order, λ_1 and λ_2 are roots of unity. Clearly, $\lambda_1, \lambda_2 \in \mathbb{R}$, and are not equal, so $\{\lambda_1, \lambda_2\} = \{-1, 1\}$. Therefore,

$$\text{trace}(\tilde{f}_{*1}^m) = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Hence, by (2.2), $L(\tilde{f}^i) = 0$ for every $i \in \mathbb{N}$. By Lemma 3.3, \tilde{f} does not have periodic points of even period strictly less than σ_f .

By Lemma 3.5 and Remark 8.13, the periodic points of odd period are contained in simple closed curves. With an argument similar to that used in the proof of Lemma 8.14, we can show that if for some $i < \sigma_f$ there exists a simple closed curve $\gamma \subset \text{Fix}(\tilde{f}^i)$, and i is minimal with this property, then i is odd and $i = \sigma_f/2$.

Then $\text{Per}(\tilde{f}) \subset \{\sigma_f, \sigma_f/2\}$ and we can complete the proof as we did in Lemma 8.14. \blacksquare

Remark 8.16 We could have proved Lemma 8.15 in the same way we proved Lemma 8.9, but the argument would have been much longer. \square

Lemma 8.17 *With Notation 8.3, let f be orientation-reversing, and let $i \in \{1, 2, \dots, k\}$ be such that f_{C_i} is a finite-order orientation-reversing component of f .*

- (1) *If $\sigma_i \leq 3$ then $n_i\sigma_i \leq 4g - 4$.*
- (2) *If $g_i = 0$, and b_i is even, or $b_i \neq \sigma_i/2$ or $k = 1$ then $n_i\sigma_i \leq 4g - 4$.*
- (3) *If $g_i = 1$ then $n_i\sigma_i \leq 4g - 4$.*
- (4) *If $g_i \geq 2$ and $b_i \neq 2$ then $n_i\sigma_i \leq 4g - 4$.*

Proof. Assume that the hypotheses of (1) hold. By Remark 8.13, σ_i is even, so $\sigma_i = 2$. Hence, by Lemma 8.4(3), $\sigma_i n_i = 2n_i \leq 4g - 4$ and the proof of (1) is complete.

Now, suppose that $g_i = 0$. By Lemma 8.4(4), $b_i \geq 3$ and, by (1) we can assume that $\sigma_i \geq 4$. If $\sigma_i \leq b_i$, then, $b_i \geq 4$ and, by Lemma 8.4(3) $2n_i \leq (b_i - 2)n_i \leq 2g - 2$. Here, by Lemma 8.4(3)

$$\sigma_i n_i \leq (b_i - 2)n_i + 2n_i \leq 4g - 4.$$

Therefore, we can assume that $\sigma_i > b_i$. By Lemma 8.14, $b_i = \sigma_i/2 + a$, for some integer a such that $a \in \{0, 2\}$. If $a = 2$, then, by Lemma 8.4(3)

$$\sigma_i n_i = 2(b_i - 2)n_i \leq 4g - 4,$$

and the result holds. Hence, we can restrict ourselves to the case $b_i = \sigma_i/2$. By Lemma 3.6, $\sigma_i/2$ is odd. Then, by hypothesis, $k = 1$. By Lemma 8.4(1), $(b_i - 2)n_i = 2g - 2$ and $b_i n_i$ is even. Since this is impossible, (2) holds.

Let us prove (3). Suppose that $g_1 = 1$. We claim that we can assume that $\sigma_i > 2b_i$. Indeed, if $\sigma_i \leq 2b_i$ then, by Lemma 8.4(1),

$$n_i\sigma_i \leq 2n_i b_i = 2(2g_i + b_i - 2)n_i \leq 2(2g - 2),$$

a contradiction. So, the claim is proved. By Lemma 8.4(3) there exists a positive integer k , such that $b_i = k\sigma_i/2$. Hence, $2b_i = k\sigma_i \geq \sigma_i$, which contradicts $\sigma_i > 2b_i$. So, the proof of (3) is complete.

Finally, we prove (4). Observe that, by Corollary 8.6, $\sigma_i \leq 4g_i + 4$. Hence, if $b_i \geq 4$, by Lemma 8.4,

$$\sigma_i n_i \leq 2(2g_i + 2)n_i \leq 2(2g_i + b_i - 2)n_i \leq 2(2g - 2),$$

as required. Now, if $b_i \in \{1, 3\}$ and h denotes the map induced by $f^{n_i}|_{C_i}$ on Σ_{g_i} , h is an orientation-reversing finite order map with a fixed point or a three periodic orbit. By Lemmas 3.5 and 3.6, $\sigma_h \in \{2, 6\}$. Observe that $\sigma_h = \sigma_i$. In particular $\sigma_i \leq 2g_i + 2$ and we can complete the proof as in the preceding paragraph. ■

Lemma 8.18 *Let $g \geq 2$ and let $f \in \mathcal{H}_g^-$ be a reducible map in standard form such that each of its components is finite-order. If the f -period of each connected component of $N(\Gamma)$ is odd then f is finite-order.*

Proof. Let A be a connected component of $N(\Gamma)$ and denote its f -period by n . Since n is odd, $f^n|_A$ is orientation-reversing. Hence, from the description of the standard form for the tubular neighborhood $N(\Gamma)$ it follows that $f^n|_A$ is conjugate to a map $\psi: \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1 \times I$ of one of the following forms.

- (1) $(z, t) \mapsto (ze^{2a\pi i}, 1 - t)$, where a is a rational number.
- (2) $(z, t) \mapsto (\bar{z}, t)$.

In particular, $f^n|_A$ is finite-order. Since $N(\Gamma)$ is f -invariant, $f|_{N(\Gamma)}$ is finite-order. Since $f|_{\Sigma_g \setminus N(\Gamma)}$ is finite-order by hypothesis, the lemma is proved. ■

Part III

Proofs of the main results

Chapter 9

Bounds for minimum periods

The aim of this chapter is to obtain bounds for the minimum periods of maps of surfaces with boundary. In Section 9.1 we apply some of the techniques developed in Chapters 2 and 7 in order to find upper bounds. In Section 9.2, we construct examples of maps with “large” minimum periods. and, by means of these examples, we prove the existence of some lower bounds. Section 9.3 is dedicated to the study of the minimum periods for classes of finite-order maps of surfaces with boundary. Theorems B, C, D and E are proved there.

9.1 Upper bounds for minimum periods

The two main results of this section are Proposition A, which was stated in the introduction (Chapter 1), and Proposition 9.5. We begin with the proof of Proposition A which is based in some of the ideas developed by Nielsen in [35], and uses Lefschetz’s Fixed-Point Theorem and some elementary algebra.

Proposition A *If $2g + b \geq 4$ then $m(\mathcal{H}_{g,b}) \leq 2g + b - 2$.*

Proof. Suppose that $f \in \mathcal{H}_{g,b}$ is a homeomorphism such that $f, f^2, \dots, f^{2g+b-3}$ are fixed-point free. In particular, $2g + b - 3 \geq 1$, so f is fixed-point free. Since $\text{Fix}(f) = \text{Fix}(f^{-1})$, f^{-1} is also fixed-point free. Then, by Theorem 2.2,

$$L(f^{-1}) = L(f) = L(f^2) = \dots = L(f^{2g+b-3}) = 0,$$

so, by (2.2),

$$\text{trace}(f_{*1}^{-1}) = \text{trace}(f_{*1}) = \text{trace}(f_{*1}^2) = \dots = \text{trace}(f_{*1}^{2g+b-3}) = 1.$$

(Recall that f_{*1} denotes the linear map induced by f on the first homology group.) The dimension of $H_1(\Sigma_{g,b}, \mathbb{Q})$, the first Betti number, is $2g+b-1 \geq 3$. So, by Proposition 7.4, $\text{trace}(f_{*1}^{2g+b-2}) \neq 1$ and, by (2.2), $L(f^{2g+b-2}) \neq 0$. Thus, by Theorem 2.2, f^{2g+b-2} has a fixed point. ■

In order to prove Proposition 9.5 we need to introduce some notation and prove an elementary lemma.

If $\text{Fix}(f^m)$ is a finite set, we denote by $P_m(f)$ the number of periodic points of f -period m , that is,

$$\text{Card}(\{x \in \text{Fix}(f^m) : x \notin \text{Fix}(f^k) \text{ for } k \in \{1, 2, \dots, m-1\}\}).$$

Then

$$\text{Card}(\text{Fix}(f^m)) = \sum_{d|m} P_d(f),$$

where $\sum_{d|m}$ denotes the sum over all positive divisors of m . Consider the *Möbius function* $\mu: \mathbb{N} \setminus \{0\} \longrightarrow \{-1, 0, 1\}$ defined by

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } k^2 | m \text{ for some } k \geq 2, \\ (-1)^r & \text{if } m = p_1 p_2 \dots p_r \text{ distinct prime factors.} \end{cases}$$

By the Möbius Inversion Formula, see [6, Proposition 13.B.3]

$$(9.1) \quad P_m(f) = \sum_{d|m} \mu(d) \text{Card}(\text{Fix}(f^{m/d})).$$

Let $f \in \mathcal{H}_g$. For each positive integer i set

$$l(f^i) = \sum_{d|i} \mu(d) L(f^{i/d}).$$

Also, for each positive integer n , we define $l_n(f) = (l(f), l(f^2), \dots, l(f^n))$.

Remark 9.1 Dold [11] proves that if Y is an ENR, and $V \subset Y$ is an open set, and $f: V \longrightarrow Y$ is a map and n is a positive integer such that $\text{Fix}(f^n)$ is compact, then n divides $l(f^n)$. This property is easily checked for the class of maps we are going to consider. As Llibre remarks in [32], these

“modified” Lefschetz numbers are interesting because, for many classes of maps, if $l(f^m) \neq 0$ then $m \in \text{Per}(f)$. In particular, this holds for finite-order maps, provided that m is strictly smaller than the order of the map (by Lemma 9.3), and for maps \tilde{f} induced by maps $f \in \mathcal{H}_{g,b}$ with “large” minimum period (by Lemma 9.2). \square

Recall that if $f \in \mathcal{H}_{g,b}$, then \tilde{f} denotes the homeomorphism induced by f on Σ_g . If the minimum period of a map f is large enough, then $l(\tilde{f}^i)$ counts the number of points of \tilde{f} -period i .

Lemma 9.2 *Let $f \in \mathcal{H}_{g,b}$. For each $i < m(f)$, $l(\tilde{f}^i) = P_i(\tilde{f})$.*

Proof. Observe that, for each $i < m(f)$, by Proposition 7.2, \tilde{f}^i is a homeomorphism with finitely many fixed points, each having index one. So, by Theorem 2.7, $L(\tilde{f}^i) = \text{Card}(\text{Fix}(\tilde{f}^i))$. Hence, by (9.1),

$$P_i(\tilde{f}) = \sum_{d|i} \mu(d) \text{Card}(\text{Fix}(\tilde{f}^d)) = \sum_{d|i} \mu(d) L(\tilde{f}^d) = l(\tilde{f}^i),$$

as desired. \blacksquare

Remark 9.3 Let i be a positive integer and let $f \in \mathcal{H}_{g,b}$. If $m(f) > i$ then, by Lemma 9.2, $l(\tilde{f}^i) \geq 0$. \square

Conjeture 9.4 Let $g \geq 2$. By Remark 9.3, if $f \in \mathcal{H}_{g,b}^+$, then $l(\tilde{f}^i) \geq 0$ for each $i < m(f)$. Hence, $m(\mathcal{H}_{g,b}^+)$ is less than or equal to the natural number m_g^+ defined as

$$\max\{n \geq 1 : \text{there exists } f \in \mathcal{H}_g^+ \text{ such that } l(f^i) \geq 0 \text{ for each } i < n\}.$$

We conjecture that $m_g^+ = 4g + 2$. Analogously, if m_g^- equals

$$\max\{n \geq 1 : \text{there exists } f \in \mathcal{H}_g^- \text{ such that } l(f^i) \geq 0 \text{ for each } i < n\},$$

then $m(\mathcal{H}_{g,b}^-) \leq m_g^-$. We conjecture that $m_g^- = 4g + (-1)^g 4$. The case $g = 2$ of these conjectures follows from some results of Chapter 10. \square

Let $f \in \mathcal{H}_{g,b}$ and for each i , let k_i denote the number of f -cycles of period i of boundary components of $\Sigma_{g,b}$. By Lemma 9.2 and Proposition 7.2, if $i < m(f)$, then \tilde{f}^i must have exactly ik_i fixed points, each with index one; hence, $l(\tilde{f}^i) = ik_i$. Now, consider the vector $(k_1, 2k_2, \dots, bk_b)$. Clearly, $\sum_{i=1}^b ik_i = b$. If i is a positive integer such that $l(\tilde{f}^i) \neq ik_i$, then $m(f) \leq i$, i.e., i is an upper bound for $m(f)$. This idea is formalized in Proposition 9.5. To state this result, we need to define a map which gives the minimum of these i .

Let n be a positive integer. Denote by $C_{b,n}$ the subset of \mathbb{N}^n consisting of all the n -tuples of the form $(k_1, 2k_2, \dots, nk_n)$ such that $\sum_{i=1}^n ik_i = b$. We define $\alpha: C_{b,n} \times \mathbb{Z}^n \longrightarrow \mathbb{N} \cup \{\infty\}$ in the following way. For each $(\bar{x}, \bar{y}) \in C_{b,n} \times \mathbb{Z}^n$,

$$\alpha(\bar{x}, \bar{y}) = \begin{cases} \infty & \text{if } \bar{x} = \bar{y}, \\ \min\{i \in \{1, 2, \dots, n\} : x_i \neq y_i\} & \text{otherwise.} \end{cases}$$

For $f \in \mathcal{H}_g$ we set

$$\gamma(f, b) = \max_{\bar{k} \in C_{b,b}} \{\alpha(\bar{k}, l_b(\tilde{f}))\}.$$

Now, for each map $f \in \mathcal{H}_{g,b}$, we can give the promised upper bound for $m(f)$, depending on $l_b(\tilde{f})$ and on the action of f on the boundary components of $\Sigma_{g,b}$. This proposition will be very useful in Chapter 10, where we will need to bound the minimum periods of maps $f \in \mathcal{H}_{g,b}$ such that the sequence $l_b(\tilde{f})$ is in a given finite subset of \mathbb{Z}^b .

Proposition 9.5 *Let $f \in \mathcal{H}_{g,b}$. For each $j \in \{1, 2, \dots, b\}$, let k_j denote the number of f -cycles of boundary components of period j . Then*

$$m(f) \leq \alpha((k_1, 2k_2, \dots, bk_b), l_b(\tilde{f})).$$

Consequently, for each $f: \Sigma_{g,b} \longrightarrow \Sigma_{g,b}$, $m(f) \leq \gamma(\tilde{f}, b)$.

Proof. Let $j \in \{1, 2, \dots, b\}$. If $m(f) > j$, by Lemma 9.2, $l(\tilde{f}^j) = P_j(\tilde{f})$. On the other hand, by Proposition 7.2, every point of \tilde{f} -period j must be a collapsed boundary component. Thus $P_j(\tilde{f}) = jk_j$. Since $(k_1, 2k_2, \dots, bk_b) \in C_{b,b}$, and

$$\alpha((k_1, 2k_2, \dots, bk_b), l_b(\tilde{f})) \geq m(f),$$

the result holds. ■

9.2 Lower bounds for minimum periods

In this section we construct examples of maps with large minimum periods on surfaces with boundary. The basic idea of these constructions consists in considering a map f of a closed surface Σ with an f -invariant set D which is a union of finitely many pairwise disjoint open disks and consider the restriction $f|_{\Sigma \setminus D}$. Clearly, $f \in \mathcal{H}_{g,b}$, where b is the number of connected components of D .

We begin with a lemma which asserts the existence of such an f -invariant set D as in the preceding paragraph, when f is finite-order and there exists an f -invariant finite set F of cardinal b .

Lemma 9.1 *Let $f: \Sigma \rightarrow \Sigma$ be a finite-order map. If $F = \{x_1, x_2, \dots, x_b\}$ is a subset of $\text{Int } \Sigma$ which is f -invariant then there exists a subset D of Σ such that*

- (1) D is f -invariant and $\text{Cl}(D) \subset \text{Int}(\Sigma)$.
- (2) *There exist b pairwise disjoint open disks D_1, D_2, \dots, D_b such that $D = \bigcup_{i=1}^b D_i$ and the center of D_i is x_i for each $i \in \{1, 2, \dots, b\}$.*

Moreover, if C is a closed subset of Σ and $F \cap C = \emptyset$ we can assume that $C \cap \text{Cl}(D) = \emptyset$.

Proof. It is a consequence of Lemmas 3.3 and 3.5. ■

Remark 9.2 Let S be a (possibly non-connected) compact orientable surface and let $f: S \rightarrow S$ be a finite-order map, that is, $f^n = \text{Id}_S$ for some positive integer n . Then Lemma 9.1 holds replacing Σ by S . □

As mentioned before, the examples we shall construct are based on finite-order maps of closed surfaces. In the following, we state a property of such maps.

Lemma 9.3 *If $f \in \mathcal{F}_g^+$ then, for each $i < \sigma_f$, $P_i(f) = l(f^i)$. In particular, i divides $l(f^i)$.*

Proof. By Lemma 3.3. $\text{Fix}(f^i)$ is a finite set consisting of points of index one. By Theorem 2.7, $\text{Card}(\text{Fix}(f^i)) = L(f^i)$, and we can now argue as in the proof of Lemma 9.2. ■

Consider a map $f \in \mathcal{F}_g^+$ and let $F \subset \Sigma_g$ be the f -invariant set consisting of all the points whose f -period is strictly less than σ_f . Let $b = \sum_{i=1}^{\sigma_f-1} l(f^i)$. By Lemma 9.3, $\text{Card}(F) = b$. So, if D is as in Lemma 9.1, then $h = f|_{\Sigma_g \setminus D} \in \mathcal{F}_{g,b}^+$ and $m(h) = \sigma_f$.

This construction applies only in the special case $b = \sum_{i=1}^{\sigma_f-1} l(f^i)$. Nevertheless, in several cases, by modifying f by means of an isotopy we can obtain another map $h \in \mathcal{H}_g^+$ with an h -invariant set F of a certain cardinal b to which we can apply Lemma 9.1 in order to find an h -invariant set consisting of b disks, and $h|_{\Sigma_g \setminus D}$ will be the desired map. This is done in Proposition 9.4.

Our next step will be to define an auxiliary map which will be used to calculate the minimum period of the map obtained by the above procedure. Let σ be a positive integer. We define $\beta: C_{b,\sigma} \times \mathbb{Z}^\sigma \longrightarrow \mathbb{N} \cup \{\infty\}$ in the following way. For each $(\bar{x}, \bar{y}) \in C_{b,\sigma} \times \mathbb{Z}^\sigma$,

$$\beta(\bar{x}, \bar{y}) = \begin{cases} \infty & \text{if } \bar{x} = \bar{y}, \\ \min(\{\frac{x_i}{i} : x_i < y_i\} \cup \{x_i - y_i : x_i > y_i\}) & \text{otherwise.} \end{cases}$$

The following result, in which we formalize the above procedure, will be applied in Proposition 9.5 to construct examples giving certain lower bounds for minimum periods.

Proposition 9.4 *Let $f \in \mathcal{F}_g^+$. Assume there exists*

$$\bar{k} = (k_1, 2k_2, \dots, \sigma_f k_{\sigma_f}) \in C_{b,\sigma_f}$$

such that $k_j = 0$ if $l(f^j) = 0$. Then there exists $h \in \mathcal{H}_{g,b}^+$ such that \tilde{h} is isotopic to f and

$$m(h) = \min\{\sigma_f, \beta(\bar{k}, l_{\sigma_f}(f))\}.$$

Furthermore, if, for each $j \in \{1, 2, \dots, \sigma_f\}$, $jk_j - l(f^j)$ is a multiple of σ_f then $\tilde{h} = f$, so h is finite-order.

Proof. By Lemma 9.3, for each $j \in \{1, 2, \dots, \sigma_f\}$,

$$P_j(f) = l(f^j) \geq \min\{l(f^j), jk_j\}.$$

Then there exists an f -invariant subset of $P_j(f)$ of cardinal $\min\{l(f^j), jk_j\}$. Let us denote this subset by F_j . Set $F = \cup_{j=1}^{\sigma_f} F_j$, an f -invariant set. Let D_1 be the f -invariant set as in Lemma 9.1 corresponding to F .

For each $j \in \{1, 2, \dots, \sigma_f\}$ we will inductively define an f -invariant subset of Σ_g , B_j , and a map $\vartheta_j: B_j \rightarrow B_j$ in the following way: If $l(f^j) \geq jk_j$, we set $B_j = \emptyset$. Otherwise, by Remark 9.3, $0 \leq l(f^j) < jk_j$. So $k_j > 0$ and, by hypothesis, $l(f^j) > 0$. By Lemma 9.3, $P_j(f) \neq \emptyset$. Consider $x \in P_j(f)$. By Lemma 3.3, there exists a disk with center x where f^j is conjugate to a rotation of order σ_f/j . Therefore, there exists an open annulus $A \subset \Sigma_g$ centered at x such that

- (1) $f^j(A) = A$.
- (2) $\bigcup_{i=0}^{j-1} f^i(A) \cap (D_1 \cup \bigcup_{i=1}^{j-1} B_j) = \emptyset$.
- (3) $f^j|_A$ is conjugate to a rotation of order σ_f/j .

Set $B_j = \bigcup_{i=0}^{j-1} f^i(A)$.

Now, we have $jk_j - l(f^j) > 0$, and we consider two cases: $jk_j - l(f^j)$ is a multiple of σ_f or not.

In the former case $jk_j - l(f^j) = a_j \sigma_f$ for some positive integer a_j , and we proceed as follows. Choose a_j disjoint f -orbits, $O_1, O_2, \dots, O_{a_j} \subset B_j$ and set $H_j = \bigcup_{i=1}^{a_j} O_i$. Let $D_2^j \subset B_j$ be a set as given by Lemma 9.1 taking $\Sigma = B_j$, $F = H_j$ and $C = \text{Cl}(D_1)$. Denote $\vartheta_j = f|_{B_j}$.

Finally, if $jk_j - l(f^j)$ is not a multiple of σ_f , set $m_j = \frac{jk_j - l(f^j)}{j}$. Since $m_j > 0$, by Lemma 9.3, m_j is a positive integer. Let $C \subset A$ be a closed annulus with non-empty interior. Then there exists a homeomorphism $\vartheta_j: B_j \rightarrow B_j$ such that

- (1) $\vartheta_j|_{\partial B_j} = f|_{\partial B_j}$ and ϑ_j is isotopic to $f|_{B_j}$ relative to ∂B_j ,
- (2) $\vartheta_j^j|_C$ is conjugate to a rotation of order m_j ,
- (3) $m(\vartheta_j) = \min\{jm_j, \sigma_f\}$.

Choose $x \in \text{Int}(C)$ and denote its ϑ_j -orbit by H_j . Then,

$$\text{Card}(H_j) = jm_j = jk_j - l(f^j).$$

(Observe that the step $j = 1$ can be done in the same way as the step $j > 1$). Let D_2^j be the set defined by Lemma 9.1 and Remark 9.2, taking S as $\bigcup_{i=1}^j f^i(C)$ and f as $\vartheta_j|_{\bigcup_{i=0}^{j-1} f^i(C)}$. It is easy to check that Lemma 9.1 holds for $\vartheta_j|_{\bigcup_{i=1}^j f^i(C)}$, so there exists a set $D_2^j \subset \bigcup_{i=1}^j f^i(C)$ satisfying (1) and (2) of Lemma 9.1.

Clearly, $D_1 \cup \bigcup_{j=1}^{\sigma_f} D_2^j$ is a union of pairwise disjoint disks. Moreover, the number of connected components of $D_1 \cup \bigcup_{j=1}^{\sigma_f} D_2^j$ is equal to $\text{Card}(F \cup \bigcup_{j=1}^{\sigma_f} H_j)$. Since

$$\text{Card}(F \cup \bigcup_{j=1}^{\sigma_f} H_j) = \sum_{j=1}^{\sigma_f} \min\{l(f^j), jk_j\} + \sum_{jk_j - l(f^j) > 0} jk_j - l(f^j) = \sum_{j=1}^{\sigma_f} jk_j = b,$$

$S = \Sigma_g \setminus (\bigcup_{j=1}^{\sigma_f} D_2^j \cup D_1)$ is a surface of genus g with b boundary components. Define $h: S \rightarrow S$ in the following way:

$$h(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{j=1}^{\sigma_f} B_j, \\ \vartheta_j(x) & \text{if } x \in B_j \text{ for some } j \in \{1, 2, \dots, \sigma_f\}. \end{cases}$$

Let us prove that $m(h) = \min\{\sigma_f, \beta(\bar{k}, l_{\sigma_f}(f))\}$. Observe that $\bigcup_{j=1}^{\sigma_f} B_j$ and $S \setminus \bigcup_{j=1}^{\sigma_f} B_j$ are h -invariant. Hence,

$$\text{Per}(h) = \text{Per}(h|_{\bigcup_{j=1}^{\sigma_f} B_j}) \cup \text{Per}(h|_{S \setminus \bigcup_{j=1}^{\sigma_f} B_j}).$$

Then

$$\min \text{Per}(h) = \min\{\min \text{Per}(h|_{\bigcup_{j=1}^{\sigma_f} B_j}), \min \text{Per}(h|_{S \setminus \bigcup_{j=1}^{\sigma_f} B_j})\}.$$

By definition of h ,

$$\min \text{Per}(h|_{\bigcup_{j=1}^{\sigma_f} B_j}) = \min \left(\{\sigma_f\} \cup \{jk_j - l(f^j) : jk_j > l(f^j), 1 \leq j < \sigma_f\} \right).$$

On the other hand, $h|_{S \setminus \bigcup_{j=1}^{\sigma_f} B_j} = f|_{S \setminus \bigcup_{j=1}^{\sigma_f} B_j}$. Hence, $j \in \text{Per}(h|_{S \setminus \bigcup_{j=1}^{\sigma_f} B_j})$ if and only if there exists $x \in P_j(f)$ such that $x \notin F_j$. By Lemma 9.3, this occurs if and only if

$$l(f^j) = P_j(f) > \text{Card}(F_j) = \min\{jk_j, l(f^j)\}.$$

Since this is equivalent to $l(f^j) > jk_j$,

$$\min \text{Per}(h|_{S \setminus \bigcup_{j=1}^{\sigma_f} B_j}) = \min\{\sigma_f\} \cup \{j : l(f^j) > jk_j\}.$$

So $m(h) = \min\{\sigma_f, \beta(\bar{k}, l_{\sigma_f}(f))\}$, as desired. ■

b	$m(\mathcal{H}_{g,b}^+) \geq$
$b \leq 2g + 2$	$b - 2$
$b = 2g + 3$	$2g$
$2g + 4 \leq b \leq 3g + 3$	$2g + 1$
$3g + 3 \leq b \leq 4g + 2$	$b - g - 2$
$4g + 2 \leq b \leq 5g + 3$	$3g$
$5g + 2 \leq b \leq 6g + 2$	$b - 2g - 2$
$6g + 2 \leq b \leq 6g + 4$	$4g$
$b = 6g + 5$	$4g + 1$
$b \geq 6g + 6$	$4g + 2$

Table 9.1: Lower bounds for $m(\mathcal{H}_{g,b}^+)$.

Let $f \in \mathcal{F}_g^+$. It is not always possible to find \bar{k} satisfying the hypotheses of Proposition 9.4. However, if f has a fixed point, we can take $\bar{k} = (b, 0, 0, \dots)$. Also, k_1 can take any value in $\{0, 1, \dots, b\}$ provided that the sum of all the k_i 's is b . This allows us, in several cases, to choose b and \bar{k} in such a way that the map $h \in \mathcal{H}_{g,b}$ given by Proposition 9.4 achieves a “large” $m(h)$ and hence, a “large” lower bound for $m(\mathcal{H}_{g,b}^+)$ is achieved. This result is stated in the following proposition which includes one of the inequalities of Theorem I.

Proposition 9.5 *For $g \geq 2$, Table 9.1 shows certain lower bounds for the values of $m(\mathcal{H}_{g,b}^+)$.*

Proof. By Corollary 6.2, there exist maps $\vartheta_1, \vartheta_2, \vartheta_3 \in \mathcal{F}_g^+$ of types

$$[4g; 0; \{1, 1, 2g\}], [2g; 0; \{1, 1, g\}] \text{ and } [4g + 2; 0; \{1, 2, 2g + 1\}]$$

respectively. Set $f_b = \vartheta_1, \vartheta_2$ or ϑ_3 as indicated in Table 9.2. Denote by σ_b the order of f_b and define $\bar{k}_b = (k_1, k_2, \dots, k_{\sigma_b})$ in the following way. If $i \notin \{1, 2, g, 2g, 2g + 1\}$ then $k_i = 0$; otherwise, k_i is defined as in Table 9.2. Denote by β_b the value of the lower bound of $m(\mathcal{H}_{g,b}^+)$ of Table 9.1. Applying Proposition 9.4 to \bar{k}_b and f_b , we obtain a map $h_b \in \mathcal{H}_{g,b}^+$ such that $m(h_b) = \beta_b$. Hence, $m(\mathcal{H}_{g,b}^+) \geq m(h_b) = \beta_b$, as desired. ■

Remark 9.6 As we shall see in Chapter 10, the lower bounds given in Proposition 9.5 are the best possible when $g = 2$ and $b \notin \{1, 2, 3, 4, 6, 8\}$. □

b	f_b	k_1	k_2	k_g	k_{2g}	k_{2g+1}	$m(h_b)$
$b \leq 2g + 2$	ϑ_1	b	0	0	0	0	$b - 2$
$b = 2g + 3$	ϑ_1	b	0	0	0	0	$2g$
$2g + 4 \leq b \leq 3g + 3$	ϑ_3	$b - 2$	2	0	0	0	$2g + 1$
$3g + 3 \leq b \leq 4g + 2$	ϑ_2	$b - g$	0	g	0	-	$b - g - 2$
$4g + 2 \leq b \leq 5g + 2$	ϑ_2	$b - g$	0	g	0	-	$3g$
$5g + 2 \leq b \leq 6g + 2$	ϑ_1	$b - 2g$	0	0	$2g$	0	$b - 2g - 2$
$6g + 2 \leq b \leq 6g + 4$	ϑ_1	$b - 2g$	0	0	$2g$	0	$4g$
$b = 6g + 5$	ϑ_3	$b - 2g - 3$	2	0	0	$2g + 1$	$4g + 1$
$b \geq 6g + 6$	ϑ_3	$b - 2g - 3$	2	0	0	$2g + 1$	$4g + 2$

Table 9.2: Values of \bar{k} and f .

Remark 9.7 Let $g \geq 2$. The results of Table 9.1 are not best possible in general. For example, assume that there exist positive integers p_1, p_2, p_3 pairwise coprime, such that each of them divides $2g + p_1 + p_2 + p_3 - 2$. Moreover, assume that there exist non-negative integers n_1, n_2, n_3 such that

$$b = n_1 p_1 + n_2 p_2 + n_3 p_3.$$

Let

$$n = \min(\{2g + p_1 + p_2 + p_3 - 2\} \cup \{p_i : n_i = 0\} \cup \{(n_i - 1)p_i : n_i \geq 2\}).$$

By Harvey's Theorem 5.14 and Proposition 9.4, there exists $f \in \mathcal{H}_{g,b}^+$ such that $m(f) = n$.

Hence, $m(\mathcal{H}_{g,b}^+)$ is bounded below by the maximum of all n obtained as above. \square

By Lemma 3.5, the fixed-point set of an iterate of an orientation-reversing map of a closed surface can contain not only isolated fixed points, but also pointwise-fixed simple closed curves. Taking this into account, we can prove the following proposition, in the same way as we proved Proposition 9.4.

Proposition 9.8 *Let $f \in \mathcal{F}_g^-$. Assume that there exists*

$$\bar{k} = (k_1, 2k_2, \dots, \sigma_f k_{\sigma_f}) \in C_{b, \sigma_f}$$

such that $k_j = 0$ if $l(f^j) = 0$. Then there exists $h \in \mathcal{H}_{g,b}^-$ such that

$$m(h) = \min\{a_f, \beta(\bar{k}, l_{\sigma_f}(\tilde{f}))\},$$

where $a_f = \sigma_f$ if $\dim(\text{Fix}(f^{\sigma_f/2})) = 0$ and $a_f = \sigma_f/2$, otherwise. Furthermore, if for each $j \in \{1, 2, \dots, \sigma_f\}$, if $jk_j - l(f^j)$ is a multiple of σ_f then $\tilde{h} = f$, so h is finite-order.

The next result is a corollary of Propositions 9.4 and 9.8

Corollary 9.9 *Let $f \in \mathcal{F}_g^+$ (resp. \mathcal{F}_g^-) be a map of type*

$$[n; 0; \{p_1, p_2, \dots, p_R\}].$$

Assume that $p_i \leq p_j$ if $i \leq j$ and that $\sum_{i=1}^s p_i = b$ for some $s \leq R$. Then there exists a map $h \in \mathcal{F}_{g,b}^+$ (resp. $\mathcal{F}_{g,b}^-$) such that

$$m(h) = \begin{cases} p_{s+1} & \text{if } s < R, \\ n & \text{if } s = R. \end{cases}$$

By Lemma 3.5, orientation-reversing finite-order maps have no isolated fixed points. Therefore, if we want to apply Proposition 9.8 to a given map $f \in \mathcal{F}_g^-$ and $\bar{k} = (k_1, 2k_2, \dots, bk_b)$, k_1 must be equal to 0. This restricts considerably the values of \bar{k} satisfying the hypotheses of Proposition 9.8. (Observe that the i -th entry of \bar{k} must be a multiple of i .) This is solved by imposing the additional requirement on f of having an invariant annulus which plays the role of the fixed point in the orientation-preserving case, allowing us to consider $k_1 \neq 0$. This idea is formalized in the following.

Proposition 9.10 *Let $f \in \mathcal{F}_g^-$. Suppose that the following hold.*

- (1) $\dim(\text{Fix}(f^{\sigma_f/2})) = 0$
- (2) *There exists an f -invariant annulus $A \subset \Sigma_g$ such that $f|_A$ is conjugate to the map $(z, t) \mapsto (ze^{\frac{2\pi i}{\sigma_f}}, 1 - t)$ on $\mathbb{S}^1 \times [0, 1]$.*
- (3) *There exists*

$$(k_1, 2k_2, \dots, \sigma_f k_{\sigma_f}) \in C_{b, \sigma_f}$$

such that for each $j \in \{2, 3, \dots, b\}$, $l(f^j) = 0$ implies $k_j = 0$.

Then there exists a map $h \in \mathcal{H}_{g,b}^-$ such that

$$m(h) = \min\{\sigma_f, \beta(\bar{k}, l_{\sigma_f}(\tilde{f}))\}.$$

Proof. Define D_1 as in the proof of Proposition 9.4. For each $j \leq \sigma_f$ we define an f -invariant set $B_j \subset \Sigma_g$ and a map $\vartheta_j: B_j \rightarrow B_j$ as follows. If $j = 1$, we set $B_j = A$ and let ϑ_1 be a map satisfying the following conditions.

- (1) $\vartheta_1|_{\partial B_1} = f|_{\partial B_1}$ and ϑ_1 is isotopic to $f|_{B_1}$ relative to ∂B_1 .
- (2) $\vartheta|_{B_1}$ is conjugate to the map $(z, t) \mapsto (ze^{\frac{2\pi i}{k_1}}, 1 - t)$ of $\mathbb{S}^1 \times [0, 1]$.
- (3) $m(\vartheta_1) = \min\{k_1, \sigma_f\}$.

Now, for each $i \in \{2, 3, \dots, \sigma_f\}$ we define B_j and ϑ_j as in the proof of Proposition 9.4, and complete the proof with arguments analogous to those used there. ■

Combining Lemma 6.2 and Proposition 9.10, we obtain a bound for the minimum period of the class of orientation reversing homeomorphisms of surfaces of odd genus.

Corollary 9.11 *If $g \geq 2$ is odd and $b \geq 6g - 6$, then there exists $f \in \mathcal{H}_g^-$ such that $h(f) = 4g - 4$. Consequently, $m(\mathcal{H}_{g,b}^-) \geq 4g - 4$ if $b \geq 6g - 6$.*

Proof. Let $f \in \mathcal{F}_g$ be as in Lemma 6.2. Define

$$\bar{k} = (k_1, k_2, \dots, k_{4g-4}) \in C_{b,4g-4}$$

by

$$k_i = \begin{cases} b - 2g + 2 & \text{if } i = 1, \\ 2g - 2 & \text{if } i = 2g - 2, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 9.10, there exists $h \in \mathcal{H}_{g,b}^-$ such that $m(h) = \min\{4g - 4, b - 2g + 2\} = 4g - 4$. ■

Another application of Proposition 9.10 yields the following.

Corollary 9.12 *Let $g \geq 2$. There exists $f \in \mathcal{H}_{g,b}^-$ such that $m(f) = \min\{b, 2g - 2\}$. Consequently, if $b \leq 2g - 2$, then $m(\mathcal{H}_{g,b}^-) \geq 2g - 2$.*

b	$m(\mathcal{H}_{g,b}^-) \geq$
$2g \leq b \leq 2g + 2$	$b - 2$
$2g + 2 \leq b \leq 2g + 4$	$2g$
$2g + 4 \leq b \leq 2g + 6$	$b - 4$
$2g + 6 \leq b \leq 4g + 4$	$2g + 2$
$4g + 4 \leq b \leq 6g + 2$	$b - 2g - 2$
$6g + 2 \leq b \leq 6g + 6$	$4g$
$6g + 6 \leq b \leq 6g + 10$	$b - 2g - 6$
$b \geq 6g + 10$	$4g + 4$

Table 9.3: Lower bounds for $m(\mathcal{H}_{g,b}^-)$.

Proof. Let $\bar{k} = (b, 0, 0, \dots) \in C_{b,2g-2}$. If g is even (resp. odd), applying Proposition 9.10 to \bar{k} and the map of Lemma 6.3 (resp. Lemma 6.2), we obtain a map $f \in \mathcal{H}_{g,b}^-$, such that $m(f) = \min\{b, 2g - 2\}$, as desired. ■

The following lemma is the analogue of Lemma 9.5 for the orientation-reversing case with the additional requirement that the genus g be even.

Proposition 9.13 *If $g \geq 2$ is even then Table 9.3 shows certain lower bounds for $m(\mathcal{H}_{g,b}^-)$.*

Proof. We prove this result by an argument analogous to that used in the proof of Proposition 9.5. In this case, we apply Proposition 9.4 to $\vartheta_1, \vartheta_2 \in \mathcal{F}_g^-$, the maps defined in Lemmas 6.4 and 6.1 respectively. Set $f_b = \vartheta_i$ as indicated in Table 9.4. Define $\bar{k}_b = (k_1, k_2, \dots, k_{\sigma_{f_b}})$ in the following way. If $i \notin \{1, 2, 4, 2g, 2g + 2\}$ then $k_i = 0$. Otherwise, k_i is defined as in Table 9.4.

Denote by β_b the value of the lower bound of $m(\mathcal{H}_{g,b}^-)$ of Table 9.3. Applying Proposition 9.8 to \bar{k}_b and f_b we obtain a map $h_b \in \mathcal{H}_{g,b}^-$ such that $m(h) = \beta_b$. Hence, $m(\mathcal{H}_{g,b}^-) \geq m(h_b) = \beta_b$ and the proof is complete. ■

Remark 9.14 As we shall see in Chapter 10 the lower bounds given in Proposition 9.13 are the best possible when $g = 2$ and $b \notin \{1, 2, 3, 4, 6, 10\}$. □

b	f_b	k_1	k_2	k_4	k_{2g}	k_{2g+2}	$m(h_b)$
$2g \leq b \leq 2g + 2$	ϑ_1	$b - 2$	2	0	0	0	$b - 2$
$2g + 2 \leq b \leq 2g + 4$	ϑ_1	$b - 2$	2	0	0	0	$2g$
$2g + 4 \leq b \leq 2g + 6$	ϑ_2	$b - 4$	0	4	0	0	$b - 4$
$2g + 6 \leq b \leq 4g + 4$	ϑ_2	$b - 4$	0	4	0	0	$2g + 2$
$4g + 4 \leq b \leq 6g + 2$	ϑ_1	$b - 2g - 2$	2	0	$2g$	0	$b - 2g - 2$
$6g + 2 \leq b \leq 6g + 6$	ϑ_1	$b - 2g - 2$	2	0	$2g$	0	$4g$
$6g + 6 \leq b \leq 6g + 10$	ϑ_2	$b - 2g - 6$	0	4	0	$2g + 2$	$b - 2g - 6$
$b \geq 6g + 10$	ϑ_2	$b - 2g - 6$	0	4	0	$2g + 2$	$4g + 4$

Table 9.4: Values of \bar{k} and f .

9.3 Minimum periods of finite-order maps

In this section we prove Theorems B, C, D and E. Almost all the statements of Theorems D and E follow from Theorems B and C. The basic idea for the proof Theorem B (resp. Theorem C) is that for each $g \geq 2$, $b \geq 3$, a pair (g, b) satisfies certain algebraic condition if and only if there exists a finite-order map $f \in \mathcal{F}_g^+$ (resp. \mathcal{F}_g^-) with exactly b periodic points of f -period strictly less than $2g + b - 2$. In such a case, Corollary 9.9 gives the desired result.

Lemma 9.1 *Let $g \geq 2$ and $b \geq 5$ and suppose that for some $f \in \mathcal{F}_{g,b}^+$ (resp. $\mathcal{F}_{g,b}^-$), $m(f) = 2g + b - 2$. Then there exists $k \in \mathcal{F}_g^+$ (resp. \mathcal{F}_g^-) of type*

$$[2g + b - 2; 0; \{p_1, p_2, \dots, p_R\}]$$

where R is a positive integer and $\sum_{i=1}^R p_i = b$.

Proof. Let $f \in \mathcal{F}_{g,b}^+$ (resp. $f \in \mathcal{F}_{g,b}^-$) be such that $m(f) = 2g + b - 2$ and let $k = \tilde{f}$. Then $k \in \mathcal{F}_g^+$ (resp. $k \in \mathcal{F}_g^-$). We will prove that the type of k is $[2g + b - 2; 0; \{p_1, p_2, \dots, p_R\}]$ where $R \geq 1$ and $\sum_{i=1}^R p_i = b$. We claim that $\sigma_k = 2g + b - 2$. Clearly, $\sigma_k = \sigma_f$. Since f is finite-order and has a periodic orbit of period $2g + b - 2$, $\sigma_k = j(2g + b - 2)$ for some positive integer j . By Theorem 3.1,

$$j = \frac{\sigma_k}{2g + b - 2} \leq \frac{4g + 2}{2g + 3} < 2.$$

Thus, $j = 1$ and the claim is proved. Hence, the type of k is

$$[2g + b - 2; 0; \{p_1, p_2, \dots, p_R\}]$$

and it only remains to check that $R \geq 1$ and $\sum_{i=1}^R p_i = b$.

Notice that every point in $\Sigma_{g,b}$ has f -period $2g+b-2$. Thus, the points of Σ_g of k -period p_1, p_2, \dots, p_R are collapsed boundary components. Let m be the number of k -orbits which are collapsed boundary components of k -period $2g+b-2$. Then m is a non-negative integer and $b = \sum_{i=1}^R p_i + m(2g+b-2)$. If $m \geq 1$ then $g \leq 1$, which is impossible. So, $m = 0$ and $b = \sum_{i=1}^R p_i$. Since $b \geq 5$, $R \geq 1$, so the proof is complete. \blacksquare

The main tool for the proof of the next result is Corollary 6.1.

Theorem B *Let $g \geq 2$. Then $m(\mathcal{F}_{g,b}^+) = 2g+b-2$ if and only if $b \in \{2, 3, 4\}$ or there exist positive integers p_1, p_2, p_3 such that they are pairwise coprime, each of them divides $2g+b-2$, and $p_1 + p_2 + p_3 = b$.*

Proof. We begin with the “only if” direction. Assume that $b = 1$. If there exists $f \in \mathcal{F}_{g,1}^+$ such that $m(f) = 2g+b-2 = 2g-1$ then, since $g \leq 2g-1$, f, f^2, \dots, f^g are fixed-point free. The single boundary component of $\Sigma_{g,1}$ is f^i -invariant for each $i \in \{1, 2, \dots, g\}$, so the iterates of the induced map \tilde{f}^i for $i \in \{1, 2, \dots, g\}$ have only one fixed point, the collapsed boundary component. By Proposition 7.2, this fixed point has index one for \tilde{f}^i for each $i \in \{1, 2, \dots, g\}$. Then, it follows from Theorem 2.7, that $L(\tilde{f}^i) = 1$ for each $i \in \{1, 2, \dots, g\}$. By Lemma 7.1, $L(\tilde{f}^{g+1}) = -g < 0$. Hence, by Lemma 3.4,

$$-g = L(\tilde{f}^{g+1}) = \chi(\Sigma_g) = 2 - 2g$$

and $\tilde{f}^{g+1} = \text{Id}$. Therefore, $g = 2$ and $f^3 = \text{Id}$. Thus, σ_f divides 3. Since $f, f^2 = f^g$ are fixed-point free, $\sigma_f = 3$. Then $\text{Per}(\tilde{f}) = \{1, 3\}$. On the other hand, \tilde{f} has only one fixed point. So, f has type $[3; 0; \{1\}]$. Since this contradicts condition (5) of Corollary 6.1, $b \geq 2$.

If $b \in \{2, 3, 4\}$ the conclusion holds. So we can assume that $b \geq 5$. By hypothesis, there exists $f \in \mathcal{F}_{g,b}^+$ such that $m(f) = 2g+b-2$. By Lemma 9.1, there exists $h \in \mathcal{F}_g^+$ of type $[2g+b-2; 0; \{p_1, p_2, \dots, p_R\}]$, for some positive integer R and $p_1, p_2, \dots, p_R \geq 1$ such that $\sum_{i=1}^R p_i = b$. Set

$$T = \frac{2g-2 + \sum_{i=1}^R p_i}{2g+b-2} - R + 2 = 3 - R.$$

Since the condition (1) of Corollary 6.1 holds, T must be even and non-negative. Then $R \in \{1, 3\}$. By the condition (5) of Corollary 6.1, $R \neq 1$. Therefore, $R = 3$ and the desired conclusion holds by Corollary 6.2.

Let us see the “if” direction. By Proposition A, $m(\mathcal{F}_{g,b}^+) \leq 2g + b - 2$. To see equality, it only remains to show that if the hypotheses hold then there exists a map $f \in \mathcal{F}_{g,b}^+$ such that $m(f) = 2g + b - 2$.

By Corollary 6.2, there exist maps $f_1, f_2, f_3 \in \mathcal{F}_g^+$ of type

$$[4g; 0; \{1, 1, 2g\}], [2g + 1; 0; \{1, 1, 1\}], [2g + 2; 0; \{1, 1, 2\}].$$

respectively. For each $b \in \{2, 3, 4\}$, applying Corollary 9.9 to f_{b-1} we can see that there exists $h_b \in \mathcal{F}_{g,b}$ such that $m(h_b) = 2g + b - 2$.

By Corollary 6.2, if $b \geq 5$, there exists a map of type

$$[2g - 2 + p_1 + p_2 + p_3; 0; \{p_1, p_2, p_3\}].$$

Since $p_1 + p_2 + p_3 = b$ the result holds by Corollary 9.9. ■

Remark 9.2 Combining a result of Gilman [17] and Corollary 6.1 we obtain the following: suppose that $f \in \mathcal{F}_g^+$, then the isotopy class of f is irreducible if and only if there exists positive integers p_1, p_2 and p_3 such that they are pairwise coprime and the type of f is $[2g + p_1 + p_2 + p_3; 0; \{p_1, p_2, p_3\}]$. Hence, if $b \geq 3$, by Theorem B, the maps $f \in \mathcal{F}_{g,b}^+$ such that $m(f) = 2g + b - 2$ are the ones for which the isotopy class of \tilde{f} is irreducible. □

Theorem D *Let $g \geq 2$. Then $m(\mathcal{H}_{g,b}^+) = 2g + b - 2$ if one of the following conditions holds.*

- (1) *There exist positive integers p_1, p_2, p_3 such that they are pairwise coprime, each of them divides $2g + b - 2$, and $p_1 + p_2 + p_3 = b$.*
- (2) *$b - 2$ divides $2g$.*
- (3) *$b - 3$ divides $2g + 1$.*
- (4) *$b \in \{1, 2, 3, 4, g + 2, 2g + 2, 2g + 4\}$.*

Proof. If (1) holds then the desired result is a consequence of Theorem B.

If $b - 2$ (resp. $b - 3$) divides $2g$, we can apply Theorem B to $p_1 = p_2 = 1$ and $p_3 = b - 2$ (resp. $p_1 = 1, p_2 = 2$ and $p_3 = b - 3$). So, if (2) holds, we are done.

Now, assume that (4) holds. If $b \in \{2, 3, 4\}$, the conclusion follows from Theorem B. If $b \in \{g + 2, 2g + 2\}$ (resp. $b \in \{2g + 4\}$) then (2) (resp. (3)) holds, so the desired result holds.

Finally, assume that $b = 1$. By Lemma 6.3 and Lemma 6.2, for each $g \geq 2$ there exists a map $f \in \mathcal{F}_{g-1}^+$ of type $[2g - 1; 0; \{1, 1, 1\}]$. Let $\{x_1, x_2, x_3\} \subset \Sigma_g$ be the f -fixed points. Set $F = \{x_1, x_2, x_3\}$ and let $D = D_1 \cup D_2 \cup D_3$ be as in Lemma 9.1. Clearly, $f(D_i) = D_i$ for each $i \in \{1, 2, 3\}$. Now, we glue to $\Sigma_g \setminus D$ an annulus A whose boundary components are ∂D_1 and ∂D_2 . Moreover, we extend $f|_{\Sigma_g \setminus D}$ to a map

$$h: (\Sigma_g \setminus D) \cup A \longrightarrow (\Sigma_g \setminus D) \cup A$$

such that $m(h) = 2g - 1$. Since $(\Sigma_g \setminus D) \cup A$ is a surface of genus g with one boundary component, the proof is complete. \blacksquare

Remark 9.3 The map constructed in the above proof for the case $b = 1$ is not finite-order because, even if f is conjugate to a rotation of order $2g - 1$ around each fixed point, it can be proved that the angles of these rotations cannot sum to 0, so $h|_A$ cannot be finite-order. \square

Theorem C *Let $g \geq 2$. Then $m(\mathcal{F}_{g,b}^-) = 2g + b - 2$ if and only if $b \in \{2, 4\}$ or one of the following conditions holds.*

- (1) g is even and there exist positive integers p_1, p_2 such that $\text{g.c.d}(p_1, p_2) = 2$, each of them divides $2g + b - 2$, and $p_1 + p_2 = b$.
- (2) g is odd, b is even, and b divides $2g - 2$.

Proof. We begin with the “only if” direction. Suppose $m(\mathcal{F}_{g,b}^-) = 2g + b - 2$. Then there exists $f \in \mathcal{F}_{g,b}^-$ such that $m(f) = 2g + b - 2$. By Lemma 6.4, b is even. If $b \in \{2, 4\}$ we are done. Hence, we can assume that $b \geq 5$. By Lemma 9.1, there exists $h \in \mathcal{F}_g^-$ of type $[2g + b - 2; 0; \{p_1, p_2, \dots, p_R\}]$, for some positive integer R , $p_1, p_2, \dots, p_R \geq 1$. Set

$$T = \frac{2g - 2 + \sum_{i=1}^R p_i}{2g + b - 2} - R + 2 = 3 - R.$$

Since condition (4) of Corollary 6.3 holds, $T \geq 1$. Hence, $R \in \{1, 2\}$ and the desired conclusion follows from Lemma 6.5.

Let us see the “if” direction. By Proposition A, $m(\mathcal{F}_{g,b}^-) \leq 2g + b - 2$. Suppose that g is odd and consider b such that either $b \in \{2, 4\}$ or b divides

$2g - 2$. By Lemma 6.5 there exists a map of type $[2g + b - 2; 0; \{b\}]$. By Lemma 9.9, there exists $f \in \mathcal{F}_{g,b}^-$ such that $m(f) = 2g + b - 2$, which proves that $m(\mathcal{F}_{g,b}^-) \leq 2g + b - 2$ in this case.

Now, assume that g is even. By Lemma 6.5, there exist maps of type $[4g; 0; \{2, 2g\}]$ and $[4g + 4; 0; \{4, 2g + 2\}]$. Applying Corollary 9.9 to these maps we obtain the desired equality for $b \in \{2, 4\}$. If $b \geq 5$, by Lemma 6.5, there exists a map of type $[2g + b - 2; 0; \{p_1, p_2\}]$. Since $p_1 + p_2 = b$ we can apply Lemma 9.9 to this map in order to obtain $f \in \mathcal{F}_{g,b}^-$ such that $m(f) = 2g + b - 2$. ■

Now we use Theorem C to prove the following result.

Theorem E *Let $g \geq 2$.*

- (1) *If b is odd then $m(\mathcal{H}_{g,b}^-) \leq b$, and equality holds if $b \leq 2g - 2$.*
- (2) *$m(\mathcal{H}_{g,b}^-) = 2g + b - 2$ if one of the following conditions holds.*
 - (i) *$b \in \{2, 4\}$.*
 - (ii) *g is odd, b is even and b divides $2g - 2$.*
 - (iii) *g is odd, $b \in \{g - 1, 2g - 2\}$.*
 - (iv) *g is even, and there exists positive integers p_1, p_2 such that each of them divides $2g + b - 2$, $\text{g.c.d}(p_1, p_2) = 2$, and $p_1 + p_2 = b$.*
 - (v) *g is even, and $b - 2$ divides $2g$.*
 - (vi) *g is even, and $b - 4$ divides $2g + 2$.*
 - (vii) *g is even, and $b \in \{g + 2, 2g + 2, 2g + 6\}$.*

Proof. We begin by proving (1). Let $f \in \mathcal{H}_{g,b}^-$. If b is odd, there is a boundary component B of $\Sigma_{g,b}$ such that $f^i(B) = B$ for some odd $i \leq b$. Since $f^i|_B$ is conjugate to an orientation-reversing homeomorphism of the circle, then, by Remark 2.5, $f^i|_B$ has a fixed point. Therefore, $m(f) \leq i \leq b$. That equality holds for $b \leq 2g - 2$ follows from Corollary 9.12.

Let us prove (2). If (i), (ii), (iii) or (iv) hold, then the result is a consequence of Theorem C. If (v) (resp. (vi)) holds, we apply Theorem C, to $p_1 = 2$ and $p_2 = b - 2$ (resp. $p_1 = 4$ and $p_2 = b - 4$) and obtain the desired conclusion.

Finally, we prove that (vii) implies (2). If $b \in \{g + 2, 2g + 2\}$ (resp. $b = 2g + 6$), then $b - 2$ divides $2g$ ($b - 4$ divides $2g + 2$), and (v) (resp. (vi)) holds, so the proof is complete. ■

Chapter 10

Homeomorphisms of surfaces of low genus

The purpose of this chapter is to study the minimum periods of maps of $\Sigma_{g,b}$ for $g \in \{0, 1, 2\}$. As we will see, the cases $g = 0$ and $g = 1$ are not hard to solve, whereas the case $g = 2$ requires more effort. In the latter case, one of the inequalities can be proved by means of the examples constructed in Chapter 9. The main tools for studying the other inequality will be Newton's equations, Lemma 7.2, and Proposition 9.5. Indeed, by means of simple calculations we will show that if a homeomorphism $h: \Sigma_2 \rightarrow \Sigma_2$ is in a “large” set then $L(h^n) < 0$ for some $n \in \{1, 2, 3, 4\}$. So, Proposition 9.5 implies that the minimum period of maps $f: \Sigma_{2,b} \rightarrow \Sigma_{2,b}$ such that \tilde{f} is in this “large” set of homeomorphisms is less than or equal to 4. For the rest of the maps $h: \Sigma_2 \rightarrow \Sigma_2$ we will show that the pair $(L(h), L(h^2))$ can take only finitely many values. For each of these possible values (c, d) , Proposition 9.5 will give an upper bound for the minimum period of the class of $f \in \mathcal{H}_{g,b}^+$ (resp. $\mathcal{H}_{g,b}^-$) inducing a map $\tilde{f} \in \mathcal{H}_g^+$ (resp. \mathcal{H}_g^-) such that $(L(\tilde{f}), L(\tilde{f}^2)) = (c, d)$.

10.1 The orientation-preserving case

This section is devoted to prove Theorem F. We begin by introducing some notation. For each $f \in \mathcal{H}_g$ and each positive integer n , let $L_n(f)$ denote the n -uple of integers,

$$(L(f), L(f^2), \dots, L(f^n))$$

Also, denote the infinite sequence

$$(L(f), L(f^2), L(f^3), \dots)$$

by $L_\infty(f)$. For $v = (v_1, \dots, v_r) \in \mathbb{Z}^r$, we write v^∞ for the sequence $(v_1, \dots, v_r, v_1, \dots, v_r, \dots) \in \mathbb{Z}^\mathbb{N}$.

Lemma 10.1 *Let $f \in \mathcal{H}_{g,b}^+$. If $L(\tilde{f}) \geq 4$ then $m(f) \leq 2g$.*

Proof. If $m(f) > 2g$, by Proposition 7.2, for each $i \in \{1, 2, \dots, 2g\}$, the fixed points of \tilde{f}^i are isolated and have index one. By Theorem 2.7, \tilde{f} has at least 4 fixed points, i.e., $\text{Card}(\tilde{f}) \geq 4$. Since $\text{Fix}(\tilde{f}) \subset \text{Fix}(\tilde{f}^i)$, for every positive integer i , $\text{Card}(\text{Fix}(\tilde{f}^i)) \geq 4$, if $i \leq 2g$. Thus, by Theorem 2.7, $L(\tilde{f}^i) \geq 4$ for each $i \in \{1, 2, \dots, 2g\}$. This contradicts Lemma 7.2, so the lemma is proved. \blacksquare

Lemma 10.2 *If $f \in \mathcal{H}_{2,b}^+$ is such that $m(f) \geq 5$ then either $L_5(\tilde{f}) = (0, 6, 12, 6, -20)$ or*

$$L_\infty(\tilde{f}) \in \{(0, 4, 6, 4, 0, -2)^\infty, (1, 3, 1, 3, 6, 1, 3, 1, 3, -2)^\infty, \\ (2, 2, 2, 6, 2, 2, 2, -2)^\infty, (3, 3, 3, 3, -2)^\infty, (2, 4, 2, 4, 2, -2)^\infty\}.$$

Proof. Fix $f \in \mathcal{H}_{2,b}^+$ such that $m(f) \geq 5$. For each positive integer i , let γ_i denote $l(\tilde{f}^i)$. Combining Newton's equations (page 22) with Lemma 7.2, and (2.3), we obtain the following system of equations,

$$\begin{aligned} p_1 + s_1 &= 0, \\ p_2 + s_1 p_1 + 2s_2 &= 0, \\ p_3 + s_1 p_2 + s_2 p_1 + 3s_3 &= 0, \\ p_4 + s_1 p_3 + s_2 p_2 + s_3 p_1 + 4s_4 &= 0, \\ s_1 &= s_3, \\ s_4 &= 1, \\ L(\tilde{f}^i) &= 2 - p_i, \text{ for } i = 1, 2, 3 \text{ and } 4, \\ \gamma_1 &= L(\tilde{f}), \\ \gamma_i &= L(\tilde{f}^i) - L(\tilde{f}), \text{ for } i = 2 \text{ and } 3, \\ \gamma_4 &= L(\tilde{f}^4) - L(\tilde{f}^2). \end{aligned}$$

γ_1	γ_3	γ_4	γ_2
0	$3(\gamma_2 - 2)$	$(-24 + 10\gamma_2 - \gamma_2^2)/2$	4, 6
1	$3(\gamma_2 - 2)/2$	$\gamma_2(2 - \gamma_2)/2$	2
2	0	$(8 - 2\gamma_2 - \gamma_2^2)/2$	0, 2
3	$-3\gamma_2/2$	$-\gamma_2(\gamma_2 + 2)/2$	0

Table 10.1: Values of γ_3 and γ_4 for $\gamma_1 \in \{0, 1, 2, 3\}$.

Solving the system for γ_3 and γ_4 we get

$$\begin{aligned}\gamma_3 &= \frac{1}{2}(-12 + 4\gamma_1 + 3\gamma_1^2 - \gamma_1^3 + 6\gamma_2 - 3\gamma_1\gamma_2), \\ \gamma_4 &= \frac{1}{2}(-24 + 26\gamma_1 + 3\gamma_1^2 - 6\gamma_1^3 + \gamma_1^4 + 10\gamma_2 - 10\gamma_1\gamma_2 + 2\gamma_1^2\gamma_2 - \gamma_2^2).\end{aligned}$$

By Lemma 10.1, $\gamma_1 \in \{0, 1, 2, 3\}$. For each of these values of γ_1 , the values of γ_3 and γ_4 as a function of γ_2 are given in the second and third columns of Table 10.1, respectively. By Lemma 10.1, $\gamma_i = l(\tilde{f}^i) \geq 0$ for each $i \in \{1, 2, 3, 4\}$. The last column of Table 10.1 gives us the values of γ_2 for which $\gamma_3 \geq 0$, and $\gamma_4 \geq 0$, for $\gamma_1 \in \{0, 1, 2, 3\}$. In other words, $L_2(\tilde{f}) \in \{(0, 4), (0, 6), (1, 3), (2, 2), (2, 4), (3, 3)\}$.

If $L_2(\tilde{f}) = (0, 6)$, then by Newton's equations (page 22) and Lemma 7.2, $L_5(\tilde{f}) = (0, 6, 12, 6, -20)$.

By Corollary 6.1, there exists $f_1 \in \mathcal{F}_2^+$ of type

$$[6; 0; \{2, 2, 3, 3\}],$$

and by Corollary 6.2, there exist maps $f_2, f_3, f_4, f_5 \in \mathcal{F}_2^+$ of types

$$[10; 0; \{1, 2, 5\}], [8; 0; \{1, 1, 4\}], [6; 0; \{1, 1, 2\}], [5; 0; \{1, 1, 1\}]$$

respectively. By Lemma 3.4, for $i = 1, 2, 3, 4, 5$, $L_\infty(f_i)$ equals

$$\begin{aligned}(0, 4, 6, 4, 0, -2)^\infty, (1, 3, 1, 3, 6, 1, 3, 1, 3, -2)^\infty, (2, 2, 2, 6, 2, 2, 2, -2)^\infty, \\ (2, 4, 2, 4, 2, -2)^\infty, (3, 3, 3, 3, -2)^\infty,\end{aligned}$$

respectively. Hence, the result follows from Remark 7.3. ■

Lemma 10.3 *If $f \in \mathcal{H}_{2,5}^+$ then $m(f) \leq 3$.*

b	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$m(\mathcal{H}_{2,b}^+)$	3	4	5	6	3	8	4	10	5	6	6	6	7	8

b	15	16	17	$b \geq 18$
$m(\mathcal{H}_{2,b}^+)$	8	8	9	10

Table 10.2: Values of $m(\mathcal{H}_{2,b}^+)$.

Proof. Assume that there exists $f \in \mathcal{H}_{2,5}^+$ such that $m(f) > 3$. Observe that

$$C_{5,5} = \{(0, 0, 0, 0, 5), (0, 2, 3, 0, 0), (1, 4, 0, 0, 0),$$

$$(1, 0, 0, 4, 0), (2, 0, 3, 0, 0), (3, 2, 0, 0, 0), (5, 0, 0, 0, 0)\}.$$

Hence, by Proposition 9.5, $l(\tilde{f}) \in \{0, 1, 2, 3, 5\}$. If $l(\tilde{f}) = 5$ then, by Proposition 9.5, $l_2(\tilde{f}) = (5, 0)$, and by Newton's equations (page 22), and Lemma 7.2, $l(\tilde{f}^3) = -21 < 0$, so, by Remark 9.3, $m(f) \leq 3$, a contradiction. So, $l(\tilde{f}) \in \{0, 1, 2, 3\}$. By Proposition 9.5, $l_3(\tilde{f}) \in B$, where

$$B = \{(0, 0, 0), (0, 2, 3), (1, 4, 0), (1, 0, 0), (2, 0, 3), (3, 2, 0)\}.$$

On the other hand, in Table 10.1, for each $l(\tilde{f}) \in \{0, 1, 2, 3\}$, the value of $l(\tilde{f}^3)$ is given in terms of $l(\tilde{f}^2)$. Using this table, a simple calculation shows that for each $h \in \mathcal{H}_2^+$, if $l(h) \in \{0, 1, 2, 3\}$ then $l_3(h) \notin B$. Since this is a contradiction, the lemma is proved. \blacksquare

Lemma 10.4 *The values of $m(\mathcal{H}_{2,b}^+)$ are given in Table 10.2.*

Proof. For each b , denote by m_b the value claimed for $m(\mathcal{H}_{2,b}^+)$ in Table 10.2.

If $b \in \{1, 2, 3, 4, 6, 8\}$, by Theorem C, $m(\mathcal{H}_{2,b}^+) = m_b$.

By Proposition 9.5, for each $b \notin \{1, 2, 3, 4, 6, 8\}$ there exists a map $f \in \mathcal{H}_{2,b}^+$ such that $m(\mathcal{H}_{2,b}^+) \geq m_b$. In particular, by Lemma 10.3, $m(\mathcal{H}_{2,5}^+) = 3$.

In Table 10.3, we list the values of $\gamma(\tilde{f}, b)$ for each f such that $l_2(\tilde{f}) \in B$, where

$$B = \{(0, 4), (1, 3), (2, 2), (2, 4), (3, 3), (0, 6)\}.$$

By Proposition 9.5, if $f \in \mathcal{H}_{g,b}^+$ is such that $l_2(\tilde{f}) \in B$ then $m(f)$ is less than or equal to the corresponding entry of Table 10.3.

$l_2(\tilde{f})/b$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	...
(0,4)	2	2	3	3	3	6	3	3	3	4	5	6	...		
(1,3)	2	3	4	10	5	5	5	5	5	6	7	8	9	10	...
(2,2)	3	8	4	4	4	4	5	6	7	8	...				
(2,4)	2	2	3	4	5	6	...								
(3,3)	2	3	4	5	...										
(0,6)	2	3	2	2	3	3	3	3	3	3	3	3	3	5	...

Table 10.3: Values of $\gamma(\tilde{f}, b)$ for $b \geq 5$ and $\tilde{f} \in B$.

Consider $f \in \mathcal{H}_{2,7}$. We claim that $m(f) \leq 4$. Indeed, if $m(f) > 4$, by Lemma 10.2, Proposition 9.5 and Table 10.3, $m(f) \leq 4$, a contradiction. Hence the claim is proved. Therefore, $4 \geq m(\mathcal{H}_{2,7}) \geq m_7 = 4$.

If $b \notin \{1, 2, 3, 4, 5, 7\}$ then $m(\mathcal{H}_{2,b}^+) \geq m_b > 4$. So, by Lemma 10.2, Proposition 9.5 and Table 10.3, $m_b \leq m(\mathcal{H}_{2,b}^+) \leq m_b$, as desired. ■

Remark 10.5 Observe that $m(\mathcal{H}_{2,b}^+) = 2.2 + b - 2 = b + 2$ if and only if $b = 1$ or $m(\mathcal{F}_{2,b}^+) = b + 2$. □

Theorem F (1)

$$m(\mathcal{H}_{0,b}^+) = \begin{cases} 1 & \text{if } b = 1, \\ \infty & \text{if } b = 2, \\ b - 2 & \text{if } b \geq 3. \end{cases}$$

(2)

$$m(\mathcal{H}_{1,b}^+) = \begin{cases} 2 & \text{if } b = 1, \\ b & \text{if } b \geq 2. \end{cases}$$

(3) Table 10.2 shows the values of $m(\mathcal{H}_{2,b}^+)$.

Proof. We will split the proof into various cases.

Case 1. $g = 0$, $b = 1$.

In this case, the result can be deduced from Brouwer's Fixed-Point Theorem and also, from Fuller's Theorem 1.2.

Case 2. $g = 0, b = 2$.

View the annulus $\Sigma_{0,2}$ as $\mathbb{S}^1 \times I$. Consider the homeomorphism $(z, \rho) \mapsto (R_\alpha(z), \rho)$, where $\alpha \in \mathbb{R}$ is irrational. It is clear that it preserves orientation and that it has no periodic points.

Case 3. $g = 0, b = 3$.

Observe that if $f \in \mathcal{H}_{0,3}^+$ then \tilde{f} is an orientation-preserving homeomorphism of the sphere Σ_0 . Since the first homology group of the sphere Σ_0 is trivial, by (2.2), $L(\tilde{f}^i) = 2$ for each positive integer i . Therefore, $l_3(\tilde{f}) = (2, 0, 0)$. Since $C_{3,3} = \{(3, 0, 0), (0, 0, 3), (1, 2, 0)\}$, the result follows from Proposition 9.5.

Case 4. $g = 0, b \geq 4$.

By Proposition A, $h_p(\Sigma_{0,b}) \leq b - 2$. The following example gives the reverse inequality. Consider a sphere with $b - 2$ holes symmetrically distributed on the equator and two more at the poles (see Figure 10.1) and take f to be rotation through an angle of $2\pi/(b - 2)$ with respect to the axis R . Clearly, f has minimum period $b - 2$.

Case 5. $g = 1, b = 1$.

By Fuller's Theorem 1.2, $h_p(\Sigma_{1,1}) \leq 2$. To complete the proof of this case we shall exhibit an example of a map on $\Sigma_{1,1}$ without fixed points.

By Lemma 6.3, there exists a map $f \in \mathcal{F}_1^+$ of type $[6; 0; \{1, 2, 3\}]$. Let x be the fixed point of f . Let D be a set as in Lemma 9.1 for $F = \{x\}$. Then $f|_{\Sigma_1 \setminus D}: \Sigma_1 \setminus D \rightarrow \Sigma_1 \setminus D$ is a homeomorphism without fixed points, as desired.

Case 6. $g = 1, b \geq 2$.

By Proposition A, $m(\mathcal{H}_{1,b}^+) \leq b$. To see that equality holds, consider a torus with b holes distributed as in Figure 10.1. Rotation through an angle of $2\pi/b$ with respect to the axis R has minimum period b , so the proof of this case is complete.

Case 7. $g = 2$.

See Lemma 10.4. ■

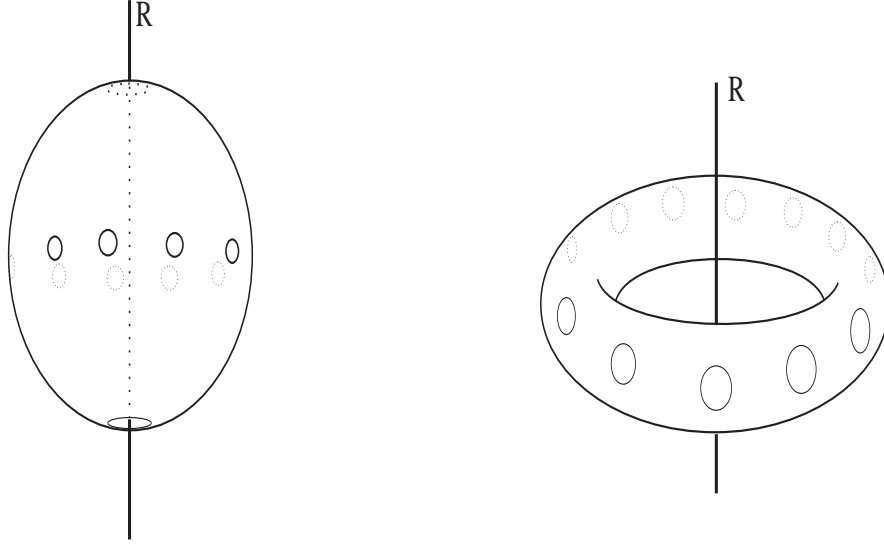


Figure 10.1: Examples for the proofs of Theorem F

10.2 The orientation-reversing case

As in the orientation-preserving case, most of this section will be devoted to studying $m(\mathcal{H}_{2,b}^-)$. One difference between the two cases is given by the following result.

Lemma 10.1 *Let i be a positive odd integer and $f \in \mathcal{H}_{g,b}^-$. If $L(\tilde{f}^i) \neq 0$ then $m(f) \leq i$.*

Proof. Assume that $m(f) > i$. Since $L(\tilde{f}^i) \neq 0$, by Theorem 2.2, \tilde{f} has a fixed point. Since $m(f) > i \geq 1$, this fixed point must be a collapsed boundary component B . Clearly, B is f^i -invariant. Since f^i is orientation-reversing, $f^i|_B$ is conjugate to an orientation-reversing homeomorphism of the circle. Thus, by Remark 2.5, $f^i|_B$ has a fixed point. Then, $m(f) \leq i$, a contradiction. ■

Lemma 10.2 *Let $f \in \mathcal{H}_{g,b}^-$. If $L(\tilde{f}^2) > 4$ then $m(f) \leq 2g$.*

b	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$m(\mathcal{H}_{2,b}^-)$	1	4	3	6	4	8	4	4	5	12	6	6	7	8

b	15	16	17	18	19	20	21	$b \geq 22$
$m(\mathcal{H}_{2,b}^-)$	8	8	8	8	9	10	11	12

Table 10.4: Values of $m(\mathcal{H}_{2,b}^-)$.

Proof. Assume that $m(f) > 2g$ and $L(\tilde{f}^2) > 4$. By Lemma 10.1, $L(\tilde{f}^i) = 0$ for each $i \in \{1, 3, 5, \dots, 2g-1\}$.

By Proposition 7.2, for each $i \in \{1, 2, \dots, 2g\}$, the fixed points of \tilde{f}^i are isolated and have index one. Since $\text{Fix}(\tilde{f}^2) \subset \text{Fix}(\tilde{f}^{2i})$, for every positive integer i , $\text{Card}(\text{Fix}(\tilde{f}^{2i})) > 4$ for each $i \in \{1, 2, \dots, g\}$. By Theorem 2.7, $4 < L(\tilde{f}^{2i})$ for each $i \in \{1, 2, \dots, g\}$. This contradicts Lemma 7.3, so the proof is complete. ■

Lemma 10.3 *The values of $m(\mathcal{H}_{2,b}^-)$ are as given in Table 10.4.*

Proof. For each b , denote by m_b the value claimed for $m(\mathcal{H}_{2,b}^-)$ in Table 10.4.

If $b \in \{1, 2, 4\}$ then conditions (1) and (2i) of Theorem E hold, so $m(\mathcal{H}_{2,b}^-) = m_b$.

By Theorem E, $m(\mathcal{H}_{2,3}^-) \leq 3$. By Lemma 6.5(2), there exists $f \in \mathcal{F}_2^-$ of type $[12; 0; \{4, 6\}]$. Let $\bar{k} \in C_{3,12}$ be such that $k_1 = 3$ and $k_i = 0$ for each $i \in \{2, 3, \dots, 12\}$. By Proposition 9.4, there exists $h \in \mathcal{H}_{2,3}^-$ such that $m(h) = 3$. Thus, $m(\mathcal{H}_{2,3}^-) = 3 = m_3$.

If $b \in \{6, 10\}$, condition (vii) of Theorem E holds, so $m(\mathcal{H}_{2,b}^-) = m_b$.

Hence, we can assume $b \geq 5$ and $b \notin \{6, 10\}$. Here, by Proposition 9.13, $m(\mathcal{H}_{2,b}^-) \geq m_b$. Suppose there exists $f \in \mathcal{H}_{2,b}^-$ such that $m(f) > m_b$. Since $m_b \geq 4$, by Lemmas 10.1 and 10.2, $l_2(\tilde{f}) \in B$, where

$$B = \{(0, 0), (0, 2), (0, 4)\}.$$

In Table 10.5, we list the values of $\gamma(\tilde{f}, b)$ for each such f . By Proposition 9.5, $m(\mathcal{H}_{2,b}^-) \leq m_b$, which contradicts our assumption and completes the proof. ■

$b \setminus l_2(\tilde{f})$	(0,0)	(0,2)	(0,4)
5	4	3	2
6	4	8	2
7	4	4	3
8	4	4	4
9	5	4	5
10	12	4	6
11	6	5	6
12	6	6	\vdots
13	6	7	
14	6	8	
15	6	8	
16	6	\vdots	
17	7		
18	8		
19	9		
20	10		
21	11		
22	12		
b	\vdots		

Table 10.5: Values of $\gamma(\tilde{f}, b)$ for $b \geq 5$ and $\tilde{f} \in B$.

Remark 10.4 As in the orientation-preserving case, $m(\mathcal{H}_{2,b}^-) = 2.2 + b - 2$ if and only if $m(\mathcal{F}_{2,b}^-) = 4 + b - 2$. \square

Theorem G (1)

$$m(\mathcal{H}_{0,b}^-) = \begin{cases} 1 & \text{if } b = 1, \\ \infty & \text{if } b = 2, \\ 2 & \text{if } b = 3, \\ b - 2 & \text{if } b \geq 4. \end{cases}$$

(2) $m(\mathcal{H}_{1,b}^-) = b - 2$.

(3) Table 10.4 shows the values of $m(\mathcal{H}_{2,b}^-)$.

Proof. We split the proof into various cases.

Case 1. $g = 0, b = 1$.

See Fuller's Theorem 1.2.

Case 2. $g = 0, b = 2$.

View the annulus $\Sigma_{0,2}$ as $\mathbb{S}^1 \times [0, 1]$ and consider the homeomorphism $(z, \rho) \mapsto (R_\alpha(z), -\rho)$, where $\alpha \in \mathbb{R}$ is irrational. It is clear that this homeomorphism reverses orientation and has no periodic points.

Case 3. $g = 0, b = 3$.

By Fuller's Theorem 1.2, $h_r(\Sigma_{0,3}) \leq 2$. We prove equality by means of an example. Consider a sphere $\Sigma_{0,3}$ with three holes symmetrically distributed on the equator, see Figure 10.2. Let $r: \Sigma_{0,3} \rightarrow \Sigma_{0,3}$ be rotation through an angle of $2\pi/3$ with respect to the axis R , and let $s: \Sigma_{0,3} \rightarrow \Sigma_{0,3}$ be reflection in the plane containing the equator. Define $f = s \circ r$. Clearly, f has no fixed points.

Case 4. $g = 0, b \geq 4$.

By Proposition A, $h_r(\mathcal{H}_{0,b}^-) \leq b - 2$. The following example shows that equality holds. Let $f: \Sigma_{0,b} \rightarrow \Sigma_{0,b}$ be the map constructed in the proof of Case $g = 0, b \geq 4$ of the proof of Theorem B, and let s be reflection in the plane containing the equator. Then $m(s \circ f) = b - 2$.

Case 5. $g = 1$.

By Lemma 9.12 and Proposition A, $h_p(\Sigma_{1,b}) \leq b$. The following example shows that equality holds. Consider a torus with b holes distributed as in Figure 10.2. Let $r: \Sigma_{1,b} \rightarrow \Sigma_{1,b}$ be rotation through an angle of $2\pi/b$ with respect to R and let $s: \Sigma_{1,b} \rightarrow \Sigma_{1,b}$ be a reflection in the plane P . Then $h(s \circ r) = b$.

Case 6. $g = 2$.

See Lemma 10.3. ■

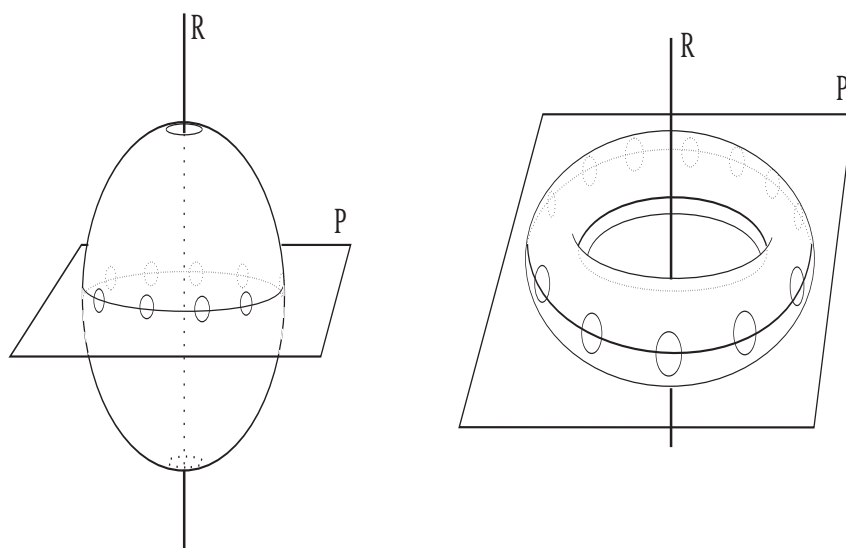


Figure 10.2: Examples for the proofs of Theorem G

Chapter 11

Proof of Theorems H and I

In this chapter we complete the proofs of Theorems H and I. Section 11.1 is dedicated to the former, and Section 11.2, to the latter.

11.1 Proof of Theorem H

This section is practically entirely devoted to proving the following theorem, which will be used to deduce the difficult inequality of Theorem H.

Theorem 11.1 *Let $g \geq 2$ and let $f: \Sigma_g \rightarrow \Sigma_g$ be an orientation preserving homeomorphism. Then there exists a positive integer m such that $m \leq 4g + 2$ and f^m has a non-empty fixed-point class of non-positive index.*

The strategy of the proof of Theorem 11.1 consists in studying fixed-point classes of non-positive index of iterates of maps in standard form. The finite-order case is trivial, so we shall concentrate on the pseudo-Anosov and reducible cases, in Propositions 11.2, 11.3, and 11.4.

Proposition 11.2 *Let $g \geq 2$ and let $f: \Sigma_g \rightarrow \Sigma_g$ be an orientation-preserving pseudo-Anosov homeomorphism in standard form. Then there exists a positive integer m such that $m \leq 4g$ and f^m has a fixed-point class of negative index.*

Proof. Let

$$\{x_1, x_2, \dots, x_k\} \subset \Sigma_g$$

be such that the set of singularities of the foliation on Σ_g is the disjoint union of the f -orbits of the x_i 's. For each $i \in \{1, 2, \dots, k\}$, let n_i and p_i be the period of x_i , and the number of prongs emanating from x_i , respectively. The Euler-Poincaré Formula (3.2) may be written in the form,

$$(11.1) \quad \sum_{i=1}^k n_i(p_i - 2) = 4(g - 1).$$

By Lemma 8.2 and Lemma 8.4(1) we can assume that $k \in \{1, 2\}$. If $k = 2$, without loss of generality, we can assume that $n_1 \leq n_2$ and, by Lemma 8.4(2), we can assume that $p_2 = 3$.

We now prove that for some positive integer m such that $m \leq 4g$, one of the parts of Lemma 8.3 applies.

Notice that if there exists a regular point of period n and $n \leq 2g$ then Lemma 8.3(6) applies for $m = 2n$. Hence, we may assume that there are no regular points of period less than or equal to $2g$. In particular,

$$(11.2) \quad \text{Per}(f) \cap \{1, 2, \dots, 2g\} \subset \begin{cases} \{n_1\} & \text{if } k = 1, \\ \{n_1, n_2\} & \text{if } k = 2. \end{cases}$$

For every positive integer h , x_1 is a fixed point of $f^{n_1 h}$. If $\text{Ind}_{f^{n_1 h}}(x_1) \neq 1$ then Lemma 8.3(1) applies. Thus, we can assume that $\text{Ind}_{f^{n_1 h}}(x_1) = 1$ for every positive integer h such that $n_1 h \leq g$. Now, we split the proof into four cases.

Case 1. $n_1 \in \{1, 2\}$.

By (11.1), $n_1 p_1 \leq 4(g-1) + 2n_1 \leq 4g$. So, Lemma 8.3(4) applies for $m = n_1 p_1$.

Case 2. $k = 2$ and $n_1, n_2 \leq g + 1$.

By (11.1),

$$n_1 p_1 + n_2 p_2 \leq 4(g - 1) + 2(n_1 + n_2) \leq 4(g - 1) + 4g + 4 = 8g.$$

Hence, $n_i p_i \leq 4g$ for some $i \in \{1, 2\}$. So, Lemma 8.3(4) applies for $m = n_i p_i$ for some $i \in \{1, 2\}$, as desired.

Case 3. $n_1 \geq g + 1$.

By (11.2), since $g + 1 \leq 2g$, $m(f) \geq g + 1$. By Theorem 2.2,

$$L(f) = L(f^2) = \dots = L(f^g) = 0.$$

By Lemmas 7.2 and 7.2, the characteristic polynomial of f_{*1} is $(x-1)^2(x^{2g-2}+1)$. Thus, the $(4g-4)$ -th power of each eigenvalue is 1. By Lemma 7.5, Lemma 8.3(2) applies with $m = 4g-4$.

Case 4. $3 \leq n_1 \leq g$ and, if $k = 2$, $n_2 > g + 1$.

Since $3 \leq n_1 \leq g \leq 2g$, by (11.2), $\text{Per}(f) \cap \{1, 2, \dots, g\} = \{n_1\}$. Moreover, the orbit of x_1 is the only periodic orbit whose f -period is smaller than g . Then the hypotheses of Proposition 7.3 hold with $n = n_1$. If for some $m \in \{1, 2, \dots, 3g-3\}$, $L(f^m) < 0$ then Lemma 8.3(2) applies, as desired. Otherwise, there exists a periodic point y of period l where $g+2 \leq l \leq 4(g-1)/3$. Since $4(g-1)/3 \leq 2g$, y is not a regular point. Therefore, $k = 2$, y is in the orbit of x_2 and $l = n_2$. Recall that $p_2 = 3$. Then $p_2 n_2 = 3l \leq 4(g-1)$. Thus, Lemma 8.3(4) applies with $m = n_2 p_2$, and the proof is complete. \blacksquare

Proposition 11.3 *Let $f: \Sigma_g \rightarrow \Sigma_g$ be an orientation-preserving reducible homeomorphism in standard form which has a pseudo-Anosov component. Then there exists a positive integer m such that $m \leq 4g-4$ and f^m has a fixed-point class of negative index.*

Proof. Let C be a pseudo-Anosov f -component. Denote by n_1 , g_1 , and b_1 the period, genus, and number of boundary components of C , respectively. Let $B \subset C$ be a p -pronged boundary component of C . Let r be the least positive integer such that $f^{rn_1}(B) = B$.

By Remark 3.2, $\sum (2 - p_s) = -pr$, where the sum is taken over all the prongs emanating from singularities s lying in $\cup_{i=1}^r f^{n_1 i}(B)$. Thus, since $p_s \geq 3$,

$$(11.3) \quad pr \leq \sum (p_s - 2),$$

where the sum is taken over all singularities s lying in C . By the Euler-Poincaré Formula (3.2) applied to $f^{n_1}|_C: C \rightarrow C$,

$$\sum (p_s - 2) = -2\chi(C) = 2(2g_1 + b_1 - 2),$$

where the sum is taken over all the singularities s of the foliation of C . Thus, by Lemma 8.4(1) and (11.3),

$$prn_1 \leq 2n_1(2g_1 + b_1 - 2) \leq 2(2g - 2) = 4g - 4.$$

Now, Lemma 8.5(1) applies and we are done. ■

Proposition 11.4 *Let $g \geq 2$ and let $f: \Sigma_g \rightarrow \Sigma_g$ be an orientation preserving reducible homeomorphism which satisfies the following conditions.*

- (1) $L(f) \neq 2$ or $L(f^2) \neq 0$.
- (2) Each of its components is finite-order.
- (3) f is not finite-order.

Then there exist an f -component C and a positive integer n such that $n \leq 4g$, $f^n|_C = \text{Id}|_C$.

Proof. With Notation 8.3, for each $i \in \{1, 2, \dots, k\}$, $f^{n_i \sigma_i}|_{C_i} = \text{Id}_{C_i}$. Hence, it suffices to show that $n_i \sigma_i \leq 4g$ for some $i \in \{1, 2, \dots, k\}$. We may assume that $g_1 \leq g_2 \leq \dots \leq g_k$. Let us split the proof into various cases.

Case 1. $k \geq 2$, $g_1 = g_2 = 0$.

By Lemma 8.4(1), $n_1(b_1 - 2) + n_2(b_2 - 2) \leq 2g - 2$. Changing subindices if necessary, by Lemma 8.4(4), we can assume that $n_1 \leq n_1(b_1 - 2) \leq g - 1$. By Lemma 8.12(1), it suffices to prove the result for $\sigma_1 = 3$. Now, $\sigma_1 n_1 = 3n_1 \leq 3g - 3$, as required.

Case 2. $k \geq 2$, $g_1 \geq 1$, $g_2 \geq 1$.

By Lemma 8.2, $n_1 g_1 + n_2 g_2 \leq g$. Changing subindices if necessary, we can assume that, $n_1 g_1 \leq g/2$. If $g_1 = 1$, by Lemma 8.12(2) we can assume that $\sigma_1 \leq 6$. If $g_1 \geq 2$, by Corollary 8.6, $\sigma_1 \leq 4g_1 + 2$. Since $n_1 \leq g/2$,

$$n_1 \sigma_1 \leq n_1(4g_1 + 2) \leq 2g + g \leq 4g,$$

and the desired conclusion holds.

Case 3. $k \geq 2$, $g_1 = 0$, $g_2 \geq 1$.

By Lemma 8.12(1), it suffices to prove the result for $b_1 = \sigma_1 = 3$. Now, if $3n_1 \leq 4g$ the result holds. Otherwise, $n_1 > \frac{4g}{3}$. By Lemma 8.4(1), $n_1 + n_2(2g_2 + b_2 - 2) \leq 2g - 2$. Then

$$n_2(2g_2 - 1) \leq n_2(2g_2 + b_2 - 2) < 2g - 2 - \frac{4g}{3} \leq \frac{2g}{3}.$$

If $g_2 = 1$, by Lemma 8.12(2), we can assume that $\sigma_1 \leq 6$. Hence, by Corollary 8.6, $\sigma_2 \leq 4g_2 + 2$, so

$$n_2 \leq n_2\sigma_2 \leq 2n_2(2g_2 + 1) = 2n_2(2g_2 - 1) + 4n_2 \leq \frac{4g}{3} + \frac{8g}{3} = 4g,$$

which completes the proof of this case.

Case 4. $k = 1, g_1 = 0$.

By Lemma 8.12(1), we can assume that $b_1 = \sigma_1 = 3$. By Corollary 8.8, the three boundary components of C_1 form a cycle under the action of f^{n_1} . By Lemma 8.5, $n_1 = 2$, and $g = 2g_1 + b_1 - 1 = 2$. So $\sigma_1 n_1 = 6 \leq 8 = 4g$, and the desired conclusion holds.

Case 5. $k = 1, g_1 \geq 1, b_1 \geq 2$.

By Lemma 8.12(2) and (3), we can assume that $b_1 = 2$. By Corollary 8.6 and Lemma 8.11(1), $\sigma_1 \in \{4g_1 + 1, 4g_1 + 2\}$. If $n_1 = 1$, by Lemma 8.4(1), $2g_1 = n_1(2g_1 + b_1 - 2) = 2g - 2$. Here,

$$n_1\sigma_1 \leq 4g_1 + 2 = 4g - 2,$$

as desired. If $n_1 \geq 2$, by Lemma 8.10(1), both boundary components are interchanged under the action of f^{n-1} . Then Lemma 8.5 gives $n_1 = 2$ and $g = 2g_1 + b_1 - 1 = 2g_1 + 1$. Hence,

$$n_1\sigma_1 \leq 2(4g_1 + 2) = 4g.$$

and we are done.

Case 6. $k = 1, g_1 \geq 1, b_1 = 1$.

If $n_1 = 1$, then by Lemma 8.4(1), $2g_1 - 1 = 2g - 2$, which is impossible. Then we can assume that $n_1 \geq 2$. By Lemma 8.11(1) and Corollary 8.6, we can assume that $\sigma_1 \in \{4g_1 + 1, 4g_1 + 2\}$. By Lemma 8.5, $n_1 = 2$.

Let A be the closed annulus connecting C_1 and $f(C_1)$. Since $f|_A$ is an orientation-preserving map which interchanges the boundary components of A , the description of standard form (page 33) shows that there exists $a \in \mathbb{Q}$ such that $f|_A$ is conjugate to the map $\psi: \mathbb{S}^1 \times [0, 1] \longrightarrow \mathbb{S}^1 \times [0, 1]$ defined by

$$(z, t) \mapsto (\bar{z}e^{a(1-2t)\pi i}, 1 - t).$$

Clearly, $\mathbb{S}^1 \times \{\frac{1}{2}\}$ is ψ -invariant. Moreover, since $\psi|_{\mathbb{S}^1 \times \{\frac{1}{2}\}}$ acts as the map $z \mapsto \bar{z}$, $\text{trace}(\psi_{*1}) = -1$. Therefore $\text{trace}((f|_A)_{*1}) = -1$, so, by (2.2), $L(f|_A) = 2$. Observe that $\Sigma_g = C_1 \cup f(C_1) \cup A$ and, since $C_1 \cap f(C_1) = \emptyset$ and $f^2(C_1) = C_1$, $f|_{C_1 \cup f(C_1)}$ does not have fixed points. Hence, there exist two open subsets of Σ_g , U and V such that $\text{Fix}(f) \subset U \subset \text{Int}(A)$, $U \cup V = \Sigma_g$ and $V \cap U \cap \text{Fix}(f) = \emptyset$. Applying Theorem 2.7 twice we obtain

$$L(f) = I(f|_U) + I(f|_V) = L(f|_A) = 2.$$

Now, observe that $f^2|_A$ is conjugate to the map $\phi: \mathbb{S}^1 \times [0, 1] \longrightarrow \mathbb{S}^1 \times [0, 1]$ defined by

$$(z, t) \mapsto (ze^{a(4t-2)\pi i}, t).$$

Also, by Lemma 8.10(1), $f^2|_{C_1}$ has no fixed points. (Observe that the restriction of f^2 to the boundary component of C_1 is a rotation of order $\sigma_1 \geq 4g_1 > 4$). Similarly, $f^2|_{f(C_1)}$ has no fixed points.

By arguments analogous to those used in the preceding paragraph, we can prove that $L(f^2) = 0$. Since this contradicts the hypotheses of the proposition, the proof of this case is complete. \blacksquare

Remark 11.5 Although in the preceding proof we use the fact that the map on the annulus A is the restriction of a reducible map in standard form, it is a simple matter to check that any orientation-preserving map of an annulus which leaves invariant each boundary component (resp. interchanges both boundary components) has Lefschetz number equal to 2 (resp. 0). \square

Proof of Theorem 11.1. Assume first that $L(f) = 2$ and $L(f^2) = 0$. In this case, by Theorem 2.2, $\text{Fix}(f) \neq \emptyset$. Since $\text{Fix}(f) \subset \text{Fix}(f^2)$, $\text{Fix}(f^2) \neq \emptyset$. Thus, if f^2 does not have a fixed-point class of index 0, by Lemma 2.7, f^2

has a fixed-point class of negative index. Since $2 \leq 4g + 2$, the result holds for this case. Now, assume that $L(f) \neq 2$ or $L(f^2) \neq 0$ and let us prove that f^m has a fixed-point class of negative index for some positive integer m such that $m \leq 4g + 2$. By Lemma 4.1 and Theorem 2.8 we can also assume that f is in standard form.

We know that there are three possibilities for f , namely, it can be of finite-order, pseudo-Anosov or reducible. If f is finite-order, then there exists a positive integer n such that $f^n = \text{Id}$. By Theorem 3.1, we can take $n \leq 4g + 2$. So, $L(f^n) = L(\text{Id}) = 2 - \text{trace}(\text{Id}) = 2 - 2g < 0$. Hence, the fixed-point class is all of Σ_g and its index is $L(f^n)$.

The remaining cases follow from Propositions 11.2, 11.3 and 11.4. ■

Theorem H *If $g \geq 2$ then $m(\mathcal{H}_{g,b}^+) \leq 4g + 2$. Moreover, if $b \geq 6g + 6$, then equality holds.*

Proof. Let $g \geq 2$. Observe that, by Corollary 9.5, $m(\mathcal{H}_{g,b}^+) \geq 4g + 2$ if $b \geq 6g + 6$. To complete the proof the theorem, it suffices to show that $m(\mathcal{H}_{g,b}^+) \leq 4g + 2$.

Let $f \in \mathcal{H}_{g,b}^+$ and consider the induced map $\tilde{f}: \Sigma_g \rightarrow \Sigma_g$. Let m be as in Theorem 11.1 for \tilde{f} . Since $m \leq 4g + 2$, it is enough to prove that $m(f) \leq m$. Consider a fixed-point class $C \subset \Sigma_g$ of \tilde{f}^m of non-positive index. If C is finite, by Lemma 2.6, the index of C with respect to \tilde{f}^m is the sum of the indices of each of its elements with respect to \tilde{f}^m . Since this sum is non-positive, at least one of its terms must be non-positive. Thus \tilde{f}^m has a fixed point of non-positive index. By Proposition 7.2, f^m has a fixed point, so $m(f) \leq m$.

If C is infinite, it contains points which are not collapsed boundary components of $\Sigma_{g,b}$. Since the existence of these fixed points of \tilde{f}^m implies the existence of fixed points of f^m , the proof of Theorem H is complete. ■

Remark 11.6 Observe that if $f \in \mathcal{H}_{g,b}^+$ is such that $m(f) > 4g$ then \tilde{f} is isotopic to a finite-order map. □

Conjeture 11.7 If $g \geq 2$, there exists $f \in \mathcal{H}_{g,b}^+$ such that \tilde{f} is isotopic to a finite-order map and $m(f) = m(\mathcal{H}_{g,b}^+)$.

This holds if $b \geq 6g + 6$ or if $g = 2$. □

11.2 Proof of Theorem I

As in the previous section, our main objective is to prove the following.

Theorem 11.1 *Let $g \geq 2$ and let $f \in \mathcal{H}_g^-$. Then there exists a positive integer m such that $m \leq 4g + (-1)^g 4$ and f^m has a fixed-point class of negative index.*

To prove Theorem 11.1 we will use the following results which study the pseudo-Anosov case, and the reducible case both with and without pseudo-Anosov components.

Proposition 11.2 *Let $g \geq 2$ be odd. If $f: \Sigma_g \rightarrow \Sigma_g$ is an orientation-reversing pseudo-Anosov map in standard form then there exists a positive integer m such that $m \leq 4g - 4$ and f^m has a fixed-point class of negative index.*

Proof. As in the proof of Proposition 11.2, we will show that one of the statements of Lemma 8.3 applies for some m such that $1 \leq m \leq 4g - 4$.

We can repeat the first part of the proof of Proposition 11.2. Combining the results obtained there with Lemma 8.3(1), (5) and (6) we can assume

- (i) $k \in \{1, 2\}$.
- (ii) If $k = 2$, then $n_1 \leq n_2$ and $p_2 = 3$.
- (iii) $\text{Ind}_{f^{n_1 h}}(x_1) = 1$ for every h such that $n_1 h \leq g$ and $n_1 h$ is even.
- (iv) There are no regular points of period less than or equal to $2g - 2$.
- (v) If for some $i \in \{1, 2\}$, $n_i \leq 2g - 2$ then n_i is even.

Now we split the proof into four cases in order to see that in each of them, one of the statements of Lemma 8.3 applies.

Case 1. $k = 2$, and $n_1, n_2 \leq g - 1$.

Here

$$n_1 p_1 + n_2 p_2 \leq 4(g - 1) + 2(n_1 + n_2) \leq 4(g - 1) + 4g - 4 = 8g - 8.$$

Therefore, we may assume $n_1 p_1 \leq 4g - 4$. By (v), Lemma 8.3(4) applies with $m = n_1 p_1$.

Case 2. $n_1 > g$.

By (iv) and Theorem 2.2,

$$L(f) = L(f^2) = \dots = L(f^g) = 0.$$

By Lemmas 7.5 and 7.2, $P(x) = (x^2 - 1)(x^{2g-2} + 1)$, so the $(4g - 4)$ -th power of the eigenvalues is 1. By Lemma 7.5, $L(f^{4g-4}) < 0$, so Lemma 8.3(2) applies.

Case 3. $n_1 = 2$, and $n_2 > g$ if $k = 2$.

By Theorem 2.2, Theorem 2.7 and (iv),

$$L(f^i) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

By (2.3), $p_i = 0$ for $1 \leq i \leq g$. Now, Newton's equations (page 22) show that $P(x) = x^{2g} - 1$, so the $(2g)$ -th power of each eigenvalue is 1. By Lemma 7.5, $L(f^{2g}) < 0$, so Lemma 8.3(2) applies with $m = 2g \leq 4g - 4$.

Case 4. $3 \leq n_1 \leq g - 1$ and $n_2 > g$ if $k = 2$.

By (v), n_1 is even and by (iv), there are no periodic regular points of period less than or equal to g because $g \leq 2g - 2$. Thus by (iii), Proposition 7.7 applies. Now, we can complete the proof of the proposition in this case as in Case 4 of the proof of Proposition 11.2. ■

Proposition 11.3 *Let $g \geq 2$ be even. If $f: \Sigma_g \rightarrow \Sigma_g$ is an orientation-reversing pseudo-Anosov map in standard form, then there exists a positive integer m such that $m \leq 4g + 4$ and f^m has a fixed-point class of negative index.*

Proof. As in the proof of Proposition 11.2, we will show that one of the statements of Lemma 8.3 applies for some positive integer m such that $m \leq 4g + 4$.

We can repeat the first part of the proof of Proposition 11.2. Combining the results obtained there with Lemma 8.3(5) and (6) we obtain

- (i) $k \in \{1, 2\}$.
- (ii) If $k = 2$, then $n_1 \leq n_2$.
- (iii) $\text{Ind}_{f^{n_1 h}}(x_1) = 1$ for every h such that $n_1 h \leq g$.

- (iv) There are no regular points of period less than or equal to $2g + 2$.
- (v) If for some $i \in \{1, 2\}$, $n_i \leq 2g + 2$ then n_i is even.

We split the proof into four cases.

Case 1. $n_1 \in \{2, 4\}$.

Here $n_1 p_1 \leq 4g - 4 + 2n_1 \leq 4g - 4 + 8 = 4g + 4$. Then Lemma 8.3(4) applies.

Case 2. $k = 2$, and $n_1, n_2 \leq g + 2$.

Here

$$n_1 p_1 + n_2 p_2 \leq 4(g - 1) + 2(n_1 + n_2) \leq 4(g - 1) + 4g + 8 = 8g + 4.$$

We can assume $n_1 p_1 \leq 4g + 2$. Since $g + 2 \leq 2g + 2$, by (v), n_1 is even. Then Lemma 8.3(4) applies.

Case 3. $n_1 \geq g + 1$.

Since $g \leq 2g + 2$, by (iv) and Theorem 2.2,

$$L(f) = L(f^2) = \dots = L(f^g) = 0.$$

Now, by Lemmas 7.5 and 7.2, $P(x) = (x^2 - 1)(x^{2g-2} - 1)$. Therefore, the $(2g - 2)$ -th power of each eigenvalue is equal to 1. By Lemma 7.5, $L(f^{2g-2}) < 0$, so Lemma 8.3(2) applies.

Case 4. $5 \leq n_1 \leq g$, and $n_2 > g + 2$, if $k = 2$.

By (v), n_1 is even. By (iv) there are no periodic regular points of period less than or equal to $g + 2$, because $g + 2 \leq 2g + 2$. By (iii), the hypotheses of Proposition 7.6 hold. Therefore there exists a positive integer m such that $m \leq 2g - 6$ and $L(f^m) < 0$. Since $2g - 6 < 4g - 4$, Lemma 8.3(2) applies in this case. ■

Proposition 11.4 *Let $f: \Sigma_g \rightarrow \Sigma_g$ be an orientation-reversing reducible map in standard form which has a pseudo-Anosov component. Then there exists a positive integer m such that $m \leq 4g - 4$ and f^m has a fixed-point class of negative index.*

Proof. We shall prove that one of the statements of Lemma 8.5 applies for some positive integer $m \leq 4g - 4$.

Let $C \subset \Sigma_g$ be a pseudo-Anosov f -component of period n_1 , genus g_1 , and b_1 boundary components. Let B be a boundary component of C . We can repeat the first part of the proof of Proposition 11.3 to prove

$$(11.4) \quad p_B n_1 r_B \leq 4g - 4,$$

where r_B is the least positive integer such that $f^{n_1 r_B}(B) = B$, and p_B is the number of prongs emanating from B . If $n_1 r_B$ is even, then, by (11.4), Lemma 8.5(1) applies. Hence we can assume that

(i) *For each boundary component B of C , $n_1 r_B$ is odd.*

If $p_B \geq 2$ for some boundary component B of C then by (11.4), $2n_1 r_B \leq p_B n_1 r_B \leq 4g - 4$. Hence, Lemma 8.5(2) applies, so we can assume that

(ii) *For every boundary component B of C , $p_B = 1$.*

Clearly, $r_B \leq b_1$. Suppose now that $g_1 \geq 1$. Then by Lemma 8.4(1),

$$2r_B n_1 \leq 2b_1 n_1 \leq 2n_1(2g_1 + b_1 - 2) \leq 4g - 4.$$

Now, by (i), Lemma 8.5(3) applies. On the other hand, if $g_1 = 0$, by Lemma 8.1. $b_1 \geq 4$. Hence, by Lemma 8.4(1), $2n_1 \leq n_1(b_1 - 2) \leq 2g - 2$. Thus we can assume

(iii) $g_1 = 0$, $2n_1 \leq n_1(b_1 - 2) \leq 2g - 2$.

Now, observe that by (ii) and Remark 3.2, $\sum_{s \in \text{Sing}(B)} (2 - p_s) = -1$ for each boundary component B of C . Thus $\sum_{s \in \text{Sing}(\partial C)} (2 - p_s) = -b_1$. By the Euler-Poincaré Formula (3.2) for $f^{n_1}|_C$

$$\sum_{s \in \text{Sing}(\text{Int } C)} (2 - p_s) - b_1 = \sum_{s \in \text{Sing}(\text{Int } C)} (2 - p_s) + \sum_{s \in \text{Sing}(\partial C)} (2 - p_s) = 2(2 - b_1).$$

So

(iv) $\sum_{s \in \text{Sing}(\text{Int } C)} (p_s - 2) = b_1 - 4$.

By (iii), $f^{2n_1}|_{C_1}$ induces an orientation-preserving map of the sphere Σ_0 . By Brouwer's theorem (or by Theorem 2.2, since $H_1(\Sigma_0)$ so $L(\tilde{f}^{2n_1}) = 2$),

this map has a fixed point x . If x is a collapsed boundary component B , then $f^{2n_1}(B) = B$. Then Lemma 8.5(1) applies for $m = 4n_1$, and, by (iii), $4n_1 \leq 4g - 4$. Thus, we can assume that x is not a collapsed boundary component. In this case, $f^{2n_1}|_C$ has a fixed point $y \in \text{Int}(C)$. If y is a regular point, Lemma 8.5(1) applies with $m = 4n_1$. As before, by (iii), we see that $m \leq 4g - 4$. Hence, we can assume that y is a singularity. Denote by p_1 the number of prongs emanating from y . By (iv),

$$2(p_1 - 2) \leq \sum_{s \in \text{Sing}(\text{Int } C)} (p_s - 2) = b_1 - 4.$$

So, $2p_1 \leq b_1$. Thus, $2p_1n_1 \leq b_1n_1 = n_1(b_1 - 2) + 2n_1 \leq 4g - 4$. Since $f^{2n_1}(y) = y$, Lemma 8.5(4) applies and the proof of the proposition is complete. ■

The following proposition is the analogue of Proposition 11.4 for the orientation-reversing case.

Proposition 11.5 *Let $g \geq 2$ and let $f \in \mathcal{H}_g^-$ be a reducible homeomorphism in standard form such that each of its components is finite-order. Then there exist an f -component C and a positive integer m such that $m \leq 4g + (-1)^g 4$ and $f^m|_C = \text{Id}_C$.*

Proof. With Notation 8.3, as in the proof of Proposition 11.4, it suffices to show that there exists $i \in \{1, 2, \dots, k\}$ such that $n_i \sigma_i \leq 4g + (-1)^g 4$. We split the proof into various cases.

Case 1. $k \geq 3$.

By Lemma 8.4(1), $\sum_{i=1}^k (2g_i + b_i - 2)n_i = 2g - 2$. Changing subindices if necessary, by Lemma 8.4(2), we can assume

$$(11.5) \quad n_1 \leq n_1(2g_1 + b_1 - 2) \leq \frac{2}{3}(g - 1).$$

Now we split the proof of this case into two subcases.

Subcase 1.1. $g_1 \in \{0, 1\}$

We claim that the result holds if $\sigma_1 \leq 2b_1$. Indeed, by (11.5)

$$n_1\sigma_1 \leq 2b_1n_1 = 2((2g_1 + b_1 - 2)n_1 + 2n_1) \leq 2\left(\frac{2}{3}(g-1) + \frac{4}{3}(g-1)\right) = 4(g-1).$$

Hence, we can assume that $\sigma_1 > 2b_1$. Now, consider $f^{n_1}|_{C_1}$. By Lemmas 8.8, 8.14 and 8.15, f^{n_1} must be orientation preserving and $g_1 = 1$. So n_1 is even. By Lemma 8.9, $\sigma_1 \leq 6$. Hence, by (11.5)

$$\sigma_1n_1 \leq 6n_1 \leq 4g - 4,$$

which completes the proof of this subcase.

Subcase 1.2. $g_1 \geq 2$

Since $b_1 \geq 1$,

$$3n_1 \leq (2g_1 - 1)n_1 \leq (2g_1 + b_1 - 2)n_1 \leq \frac{2}{3}(g-1).$$

Thus, by Corollary 8.6, $\sigma_1 \leq 4g_1 + 4$ and, since $b_1 \geq 1$, by (11.5)

$$n_1\sigma_1 \leq n_1(4g_1 + 4) = 2(n_1(2g_1 - 1) + 3n_1) \leq 2\left(\frac{2}{3}(g-1) + \frac{2}{3}(g-1)\right) \leq 4(g-1),$$

as desired.

Case 2. $k = 2$.

We consider three subcases.

Subcase 2.1. $g_1, g_2 \geq 1$.

Since, by Lemma 8.2, $n_1g_1 + n_2g_2 \leq g$, we can assume without loss of generality that $n_1g_1 \leq g/2$.

If $g_1 = 1$ then by Lemmas 8.17(3) and 8.12(2), the result holds if n_1 is odd or $\sigma_1 \geq 7$. Now, suppose that n_1 is even and $\sigma_1 \leq 6$. In particular, $2 \leq n_1\sigma_1 \leq g/2$. Then $g \geq 4$ and

$$n_1\sigma_1 \leq 6n_1 \leq 3g \leq 4g - 4,$$

as desired. If $g_1 \geq 2$, then $2 \leq n_1g_1 \leq g/2$. Hence, $g \geq 4$, and by Corollary 8.6,

$$n_1\sigma_1 \leq n_1(4g_1 + 4) \leq 2g + g \leq 4g - 4,$$

and the proof of this subcase is complete.

Subcase 2.2. $g_1 = g_2 = 0$.

By Lemma 8.4(1),

$$(11.6) \quad n_1(b_1 - 2) + n_2(b_2 - 2) = 2g - 2.$$

If n_1 and n_2 are even, then, by (11.6), $n_i(b_i - 2) \leq g - 1$ for some $i \in \{1, 2\}$ and the result follows from Lemma 8.12(1).

If n_1 and n_2 are odd, by Lemma 8.17(2), we can assume that, for each $i \in \{1, 2\}$, $b_i = \sigma_i/2$ and b_i is odd. By Lemma 8.14, the boundary components if C_i form a cycle under the action of f^{n_i} . Therefore, they remain fixed under the action of $f^{n_i b_i}$. Hence, the f -period of each connected component of $N(\Gamma)$ is odd. By Lemma 8.18, f is finite-order. Then we take the whole Σ_g as C_1 and the result holds by Theorem 3.1.

To complete the proof, changing subindices if necessary, we can assume that n_1 is odd and n_2 is even. Then, by (11.6), b_1 must be even and the result follows from Lemma 8.17(2).

Subcase 2.3. $g_1 = 0$, $g_2 \geq 1$.

By Lemma 8.17(2) and (4), we can assume that $n_2(2g_2 + b_2 - 2)$ is even. Then, $n_1(b_1 - 2)$ is even. If n_1 is odd, the result hold by Lemma 8.17(2). If n_1 is even, by Lemma 8.12(1), we can assume that $b_1 = 3$ and $n_1 > \frac{4}{3}(g - 1)$. By Lemma 8.4(1),

$$n_2(2g_2 + b_2 - 2) = 2g - 2 - n_1(b_1 - 2) < \frac{2}{3}(g - 1).$$

and we can complete the proof as we did in Case 1.

Case 3. $k = 1$, $g_1 = 0$.

If $n_1 = 1$ the desired conclusion follows from Corollary 8.6. Therefore, we can assume that $n_1 > 1$. By Lemma 8.4(1), $n_1(b_1 - 2) = 2g - 2$.

Assume that n_1 is even. By Lemma 8.12(1), we can assume that $\sigma_1 = b_1 = 3$ and the three boundary components of C_1 form a cycle under the action of f^{n_1} . Then we can apply Lemma 8.5 to obtain $n_1 = 2$ and $g = 2g_1 + b_1 - 1 = 2$. Hence, $\sigma_1 n_1 = 6 \leq 8 = 4g + (-1)^g 4$.

If n_1 is odd, since $(b_1 - 2)n_1 = 2g - 2$, b_1 is even and the result follows from Lemma 8.17(2).

Case 4. $k = 1, g_1 \geq 1$.

Again we consider three subcases.

Subcase 4.1. $b_1 = 1$.

By Lemma 8.5, $n_1 = 2$ and $g = 2g_1 + b_1 - 1 = 2g_1$. In particular, g and n_1 are even. Then, by Corollary 8.6, $\sigma_1 n_1 \leq (4g_1 + 2)n_1 = 4g + (-1)^{g_1} 4$.

Subcase 4.2. $b_1 = 2$.

If the boundary components of C_1 form a cycle under the action of f^{n_1} then, by Lemma 8.5, $n_1 = 2$, so C_1 and $f(C_1)$ are connected by two annuli A_1 and A_2 . Since $f(A_1) = A_1$ or A_2 , $f^2(A_1) = A_1$ and $f^2(A_2) = A_2$. Then the two boundary components of C_1 cannot form a cycle under the action of f^{n_1} , which contradicts our assumption. Therefore, both boundary components of C_1 are mapped to themselves under the action of f^{n_1} . By Lemma 8.11(3), we can assume that $\sigma_1 \geq 3$. By Lemma 8.10(2), $f^{n_1}|_{C_1}$ must be orientation preserving, so n_1 is even and, by Lemma 8.10(1), $\sigma_1 \leq 4g_1$. Then $n_1 \sigma_1 < 4g_1 n_1 = 4g - 4$.

Subcase 4.3. $b_1 = 3$.

If $g_1 = 1$ the result follows from Lemma 8.12(2). Now, let $g_1 \geq 2$. By Lemma 8.4(1),

$$n_1(2g_1 + 1) = n_1(2g_1 + b_1 - 2) = 2g - 2.$$

This implies that n_1 is even, and Lemma 8.12(1) and (2) completes the proof. ■

Proof of Theorem 11.1. By Lemma 4.1 and Theorem 2.8 we can assume that f is in standard form.

We know that there are three possibilities for f , namely, it can be of finite-order, pseudo-Anosov or reducible. If f is finite-order, then there exists a positive integer n such that $f^n = \text{Id}$. By Theorem 3.1, we can take $n \leq 4g + (-1)^{g_1} 4$, so by (2.2), $L(f^n) = L(\text{Id}) = 2 - \text{trace}(\text{Id}) = 2 - 2g < 0$. In this case, the fixed-point class is all of Σ_g and its index is $L(f^n)$.

The remaining cases are consequences of Propositions 11.2, 11.3, 11.4 and 11.5. ■

Theorem I *Let $g \geq 2$. Then $m(\mathcal{H}_{g,b}^-) \leq 4g + (-1)^g 4$ and equality holds if $b \geq 6g + 2 + (-1)^g 8$.*

Proof. By Proposition 9.13, $m(\mathcal{H}_{g,b}^+) \geq 4g + 4$ if $b \geq 6g + 10$ and g is even. By Corollary 9.11, $m(\mathcal{H}_{g,b}^+) \geq 4g - 4$ and $b \geq 6g - 6$ if g is odd. Hence, to complete the proof of the theorem, it suffices to show that $m(\mathcal{H}_{g,b}^+) \leq 4g + (-1)^g 4$. Now, we can complete the proof as we did for Theorem H. ■

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