Applying (2) with $\{x, y\} = \{ f_r, \alpha^r/\sqrt{5}\}$ gives

$$\left| f_r^{1/k} - \frac{\alpha^{r/k}}{5^{1/(2k)}} \right| \le \frac{\alpha^{-r}}{k\sqrt{5}} \min\left(f_r, \alpha^r/\sqrt{5}\right)^{1/k-1}.$$

When $r \ge 5$, applying (1) yields

$$\left| f_r^{1/k} - \frac{\alpha^{r/k}}{5^{1/(2k)}} \right| \le \frac{1.01}{k5^{1/(2k)}}\alpha^{r/k-2r}. \tag{4}$$

Since $\alpha^2 = \alpha + 1$, we have $\alpha^{n/k} = \alpha^{(n-k)/k} + \alpha^{(n-2k)/k}$. Thus

$$\left| f_n^{1/k} - \left(f_{n-k}^{1/k} + f_{n-2k}^{1/k}\right)\right| \le \left| f_n^{1/k} - \frac{\alpha^{n/k}}{5^{1/(2k)}}\right| + \left| f_{n-k}^{1/k} - \frac{\alpha^{(n-k)/k}}{5^{1/(2k)}}\right| + \left| f_{n-2k}^{1/k} - \frac{\alpha^{(n-2k)/k}}{5^{1/(2k)}}\right|.$$

Since $n \ge 12$ and $n \ge 4k-3$, we have $n-2k \ge 5$. We may therefore apply (4) with $r = n$, $r = n-k$, and $r = n-2k$ to conclude that

$$\begin{aligned} \left| f_n^{1/k} - \left(f_{n-k}^{1/k} + f_{n-2k}^{1/k}\right)\right| &\le \frac{1.01}{k5^{1/(2k)}}\alpha^{n/k-n}\alpha^{4k-n-2}\left(1 + \alpha^{1-2k} + \alpha^{2-4k}\right) \\ &\le \frac{2.12}{k5^{1/(2k)}}\alpha^{n/k-n}, \end{aligned} \tag{5}$$

where the last inequality uses $n \ge 4k-3$ and $k \ge 2$.

Comparing (3) and (5), we see that $\sqrt[k]{f_{n-k}} + \sqrt[k]{f_{n-2k}}$ is closer to $\sqrt[k]{f_n}$ than $\sqrt[k]{f_n}$ is to an integer.

Editorial comment. It is fairly easy to prove that for each fixed k, the assertion is true for sufficiently large n.

A Divisibility Result for a Combination of mth Powers

10770 [1999, 963]. *Proposed by Călin Popescu, Louvain-la-Neuve, Belgium.* Suppose that m and n are integers with $1 < m < \phi(m) + n$, where $\phi(m)$ is the number of elements in $\{1, 2, \dots, m\}$ that are relatively prime to m. Show that $\sum_{i=1}^n (-1)^i \binom{n}{i} i^m$ is divisible by m.

Solution I by Jim Vandergriff, Austin Peay State University, Clarksville, TN. We have $n > k$, where $k = m - \phi(m)$. A generalization of Euler's generalization of Fermat's Little Theorem states that $a^m \equiv a^{m-\phi(m)} \pmod m$ for every integer a (I. Niven and H. Zuckerman, *An Introduction to the Theory of Numbers*, John Wiley & Sons, 4th ed., p. 51, problem 22). Thus

$$\sum_{i=1}^{n}(-1)^i \binom{n}{i} i^m \equiv \sum_{i=1}^{n}(-1)^i \binom{n}{i} i^k \pmod m.$$

We now show that the sum on the right is 0 when $k < n$. Consider the generating function $f_0(x) = \sum \binom{n}{i} x^i$. Letting $f_r(x) = x f_{r-1}'(x)$ for $r > 0$ yields $f_r(x) = \sum i^r \binom{n}{i} x^i$. Also $f_0(x) = (x+1)^n$, and inductively each f_r has $(x+1)^{n-r}$ as a factor as long as $r < n$. Setting $x = -1$ in f_k yields the result.

Solution II by Jim Delany, California Polytechnic Institute, San Luis Obispo, CA. Letting $i = n - j$ in the given sum turns it into $(-1)^n$ times the familiar inclusion-exclusion sum for the number $T(m, n)$ of surjections from an m-element set to an n-element set. We view such a function as placing elements of $\{1, \dots, n\}$ in m positions around a circle. We partition these into equivalence classes with m functions in each class, thereby showing that $T(m, n)$ is a multiple of m.

The condition $1 < m < \phi(m) + n$ implies that n is greater than every proper divisor of m. To see this, note that the largest proper divisor of m is m/q, where q is the smallest prime divisor of m. Since $\phi(m) = m\prod_{p|m}(1-1/p) \le m(1-1/q)$, we obtain $m/q \le m - \phi(m) < n$.

This implies that the m rotations of each circular arrangement are distinct. If some rotation of an arrangement leaves it unchanged, then its positions fall into sets on which the label is constant. The number of these sets divides m, but the requirement of surjectivity implies that there must be at least n such sets. Since n exceeds every proper divisor of m, the arrangements fall into rotational classes of size m.

Solved also by S. Amghibech (France), J. C. Binz (Switzerland), S. Cautis (Canada), R. J. Chapman (U. K.), W. Chu (France), J. Delany, J. Flowers, A. E. Gurel (Turkey), N. Komanda, J. H. Lindsey II, O. P. Lossers (The Netherlands), A. Marini (France), H.-J. Seiffert (Germany), S. Siciliano (Italy), and the proposer.

Three-dimensional Lattice Walks in the Upper Half-Space

10795 [2000, 367]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* A *3-dimensional lattice walk* of length n takes n successive unit steps, each in one of the six coordinate directions. How many 3-dimensional lattice walks of length n are there that begin at the origin and never go below the horizontal plane?

Solution by Jim Brawner, Armstrong Atlantic State University, Savannah, GA. There are $\sum_{k=0}^{n}\binom{n}{k}\binom{k}{\lceil k/2\rceil}4^{n-k}$ such walks.

Among these walks, consider those with exactly k vertical steps. Each non-vertical step can be in any of the four non-vertical directions, giving 4^{n-k} possibilities for the directions of these steps.

Since the walk never goes below the horizontal plane, the list of k vertical steps must be "up-dominated", meaning that every initial segment of steps has at least as many upward steps as downward steps. Up-dominated binary lists with r upward steps and $k-r$ downward steps were first counted by A. D. André in solving Bertrand's Ballot Problem; there are $\binom{k}{r}-\binom{k}{r+1}$ of them. Note that $r \ge \lceil k/2\rceil$. Summing over the possible values of r yields a telescoping sum for the number of up-dominated lists of length k; there are $\binom{k}{\lceil k/2\rceil}$ of them.

Finally, there are $\binom{n}{k}$ ways in which the vertical (and non-vertical) steps can be distributed among the n steps of the walk. Summing over all possible values for the number of vertical steps yields the answer.

Editorial comment. Solvers used various methods to solve this problem. These yielded many alternative expressions for the answer. These included

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k+1}{k}2^{n-k}, \qquad 6^n - \sum_{k=0}^{n-1}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}C_j4^{k-2j}6^{n-k-1},$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}C_k6^{n-k}, \quad \text{and} \quad 2^{-n-1}\left(3^n(5n+2)C_n - \sum_{k=0}^{n-1}3^k(k+1)C_kC_{n-k}\right),$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.

Solved also by D. Beckwith, O. Byer and C. Cooley, R. J. Chapman (U. K.), R. DiSario, GCHQ Problems Group (U. K.), J.-P. Grivaux (France), O. P. Lossers (The Netherlands), R. F. McCoart, A. Nkwanta, E. Purdy, R. Stong, B. Willis, L. Zhou, and the proposer.

Incognito Hypergeometrics

10836 [2000, 864]. *Proposed by Jon A. Wellner, University of Washington, Seattle, WA.* Show that

$$4\sqrt{\pi}(1-x^2)^{3/2}\sum_{k=0}^{\infty}x^{2k}\frac{\Gamma(k+1)}{\Gamma(k+1/2)} = 4+\sum_{k=1}^{\infty}x^{2k}(2k)!\left(\sum_{j=0}^{k}(-1)^j\frac{2^{k-j+1}}{j!2^j}\frac{(k-j)!}{(2k-2j)!}\right)^2$$