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NOTES

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A Simple Congruence modulo p

Winfried Kohnen

Congruences for prime numbers p have always been of great interest. Examples include Fermat's Little Theorem ($n^p \equiv n \pmod{p}$) or Wilson's theorem ($(p-1)! \equiv -1 \pmod{p}$). In the following we consider the congruence relation modulo p extended to the ring of rational numbers with denominators not divisible by p . For such fractions $m/n \equiv r/s \pmod{p}$ if and only if $ms \equiv nr \pmod{p}$, and the residue class of m/n is the residue class of m times the inverse of the residue class of n in \mathbf{Z}_p .

The purpose of this note is to state and prove the following result.

Theorem. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^{k-1}}{k} \pmod{p}. \quad (1)$$

Proof: First note the identity

$$\sum_{k=1}^N \frac{1}{k} (1-X)^k = \sum_{k=1}^N \frac{(-1)^k}{k} \binom{N}{k} (X^k - 1) \quad (N \in \mathbf{N}, x \in \mathbf{R}). \quad (2)$$

Indeed, the derivative of the left-hand side of (2) is

$$-\sum_{k=1}^N (1-X)^{k-1} = -\frac{1 - (1-X)^N}{1 - (1-X)} = \frac{(1-X)^N - 1}{X},$$

while the derivative of the right-hand side is

$$\sum_{k=1}^N (-1)^k \binom{N}{k} X^{k-1}.$$

Hence the derivative of both sides are equal. Also (2) is true for $X = 1$.

In (2) we set $N = p - 1$ and $X = -1$. From $p - k \equiv -k \pmod{p}$, we deduce

$$\binom{p-1}{k} = \frac{(p-1) \cdots (p-k)}{k!} \equiv (-1)^k \pmod{p}$$

and

$$\sum_{k=1}^{p-1} \frac{1}{k} = \sum_{k=1}^{p-1} \frac{1}{p-k} \equiv -\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p},$$

and thus equation (2) simplifies to

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \pmod{p}. \quad (3)$$

In the sum on the left we replace k by $p - k \equiv -k \pmod{p}$ and use Fermat's Little Theorem to obtain

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2^p \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv -2 \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \pmod{p}.$$

The sum on the right of (3) we rewrite as

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} + \sum_{k=1}^{(p-1)/2} \frac{(-1)^{p-k}}{p-k} \equiv 2 \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} \pmod{p}.$$

This proves (1).

In the literature, congruences of a type similar to (1) can be found; however, in general they are of a much deeper nature. For example, in [1] with the help of properties of the Pell sequence $((1 + \sqrt{2})^n)_{n \in \mathbb{N}}$ it is shown that

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{\lfloor 3p/4 \rfloor} \frac{(-1)^{k-1}}{k} \pmod{p}. \quad (4)$$

It seems unlikely that (4) can be proved with the simple approach we have used here.

REFERENCE

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A Geometrical Method for Finding an Explicit Formula for the Greatest Common Divisor

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This note presents an explicit formula for the greatest common divisor (g.c.d.) of two integers derived using a simple geometrical argument.

In [1], chapter 3, an expression was deduced, from which one can easily obtain a formula for the g.c.d. as a particular case. However, the derivation of that expression is very tiring and lengthy.