67.2 A Combinatorial Approach to Goldbach's Conjecture
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Hence

$$f(x) = 2\sin\underbrace{\left(\frac{\pi}{4} + \frac{\sin x - \cos x}{2}\right)}_{\text{between 0 and } \frac{\pi}{2}} \sin\underbrace{\left(\frac{\pi}{4} - \frac{\sin x + \cos x}{2}\right)}_{\text{between 0 and } \frac{\pi}{2}}$$

and $f(x) > 0$ for all real x.

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67.2 A combinatorial approach to Goldbach's conjecture

In 1742 Goldbach suggested in a letter to Euler that every even integer greater than 4 is the sum of two odd primes. Although many attempts have been made to establish this conjecture, a proof is still awaited. The "circle method" of Hardy and Littlewood [1] showed that the method was tractable on the assumption of the Riemann hypothesis, which is again topical. By using a combinatorial approach which can be further extended by the interested reader, it is demonstrated here that it is highly unlikely that Goldbach's conjecture is false.

Our approach is best introduced by an example. Consider 18 and tabulate all the ways of adding two odd integers greater than 1 to give 18:

3	15
5	13
7	11
9	9.

Assume that there are h_1 primes in the first column and h_2 primes in the second (3 and 2 in our example). If a prime in the first column is adjacent to a prime in the second, then Goldbach's conjecture works for this particular number (7 + 11 in our case).

Given two adjacent columns of r entries, in how many ways is it possible to choose h_1 in the first column and h_2 in the second so that no two of the chosen ones are adjacent? Having chosen the h_1 the other h_2 may be chosen in $\binom{r-h_1}{h_2}$ ways, and so the total number of ways of choosing the h_1 and h_2 *without* adjacent choices is

$$\binom{r}{h_1}\binom{r-h_1}{h_2}.$$

On the other hand the number of ways of freely choosing h_1 in the first column and h_2 in the second is

$$\binom{r}{h_1}\binom{r}{h_2}.$$

So if, in our set-up, there are r numbers in each column, the probability of violating Goldbach's conjecture is

$$\binom{r}{h_1}\binom{r-h_1}{h_2}\bigg/\binom{r}{h_1}\binom{r}{h_2}$$

i.e.

$$\binom{r-h_1}{h_2}\bigg/\binom{r}{h_2}.$$

Goldbach's conjecture has been verified by a computer study to be correct for all even integers upto 10^8. And, for large numbers, $h_1 \approx h_2$, as indicated by Hardy and Wright [2]. For example, the number of primes less than 5000 is 667 while the number of primes between 5000 and 10 000 is 571. So, for large r, the probability of violating Goldbach's conjecture is given by approximately

$$\binom{r-h}{h}\bigg/\binom{r}{h} = \frac{(r-h)(r-h-1)\dots(r-2h+1)}{r(r-1)\dots(r-h+1)}$$
$$= \left(1-\frac{h}{r}\right)\left(1-\frac{h}{r-1}\right)\dots\left(1-\frac{h}{r-h+1}\right)$$
$$=< \left(1-\frac{h}{r}\right)^h.$$

For example the number of primes less than 10^6 is 78 498 and so considering 10^6 in this way gives $r \approx 250\,000$ and $h \approx 39\,250$. So the probability in this case is less than

$$\left(1-\frac{39\,250}{250\,000}\right)^{39\,250} \approx 10^{-2911}$$

This is an exceedingly low probability, so low in fact that only a pure mathematician would not consider the event to be impossible!

Further investigations suggest themselves as exercises. For instance, by utilising the results of Rosser and Schoenfeld [4], that if $\pi(x)$ is the number of primes less than x, then

$$x/\log x < \pi(x) \quad \text{for} \quad x > 17$$

and

$$3x/(5 \log x) < \pi(2x) - \pi(x) \quad \text{for} \quad x \geq 21,$$

one can test the claim that the number of primes less than x is approximately the same as the number of primes between x and $2x$ as x increases. Finer estimates of (computer- or calculator-generated) numerical data would also permit tabulations of the probability.

Further relevant exercises can be obtained by extending some of the ideas in Mirsky's historical survey [3] of the problem.

References

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4. J. Barkley Rosser and Lowell Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962).

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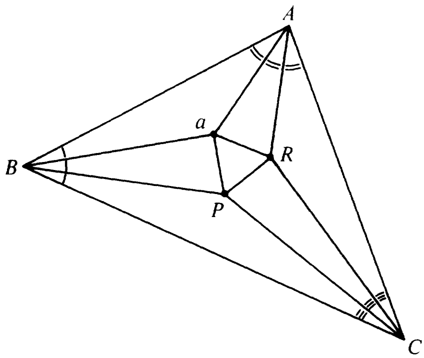
67.3 More of Morley

Morley's theorem says that given any triangle ABC, the triangle formed by joining the intersections of adjacent trisectors of the angles of ABC is equilateral.



We'll prove this result using the most elementary geometry.

Consider any triangle ABC, draw in the trisectors of angles B and C and then delete from the figure AB and AC so as to leave triangle BCD with base angle bisectors meeting at P: