A Proof of the Prime Number Theorem
Author(s): E. Grosswald
Source: *The American Mathematical Monthly*, Vol. 71, No. 7 (Aug. - Sep., 1964), pp. 736-743
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2310887
Accessed: 24/03/2010 17:51

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/action/showPublisher?publisherCode=maa.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association of America is collaborating with JSTOR to digitize, preserve and extend access to *The American Mathematical Monthly.*

http://www.jstor.org

$$0 = ((A - s_a I)X_{m_a-1}, (A - s_a I)X_{m_a-1}) + (m_a - 1)^2(X_{m_a}, X_{m_a}).$$

From this we can conclude that for $m_a > 1$

$$(A - s_a I)X_{m_a-1} = 0.$$

By repeating this argument successively in (17) we find that $X_b = 0$ for $b > 1$. This shows that the solution in (16) contains only one nonvanishing element. This implies that all $B_{a,b}$ in (8) vanish for $b > 1$. Therefore the Jordan canonical form of A is diagonal.

This last result can be strengthened considerably. We require the following lemma; its proof is trivial.

LEMMA. *If X_a and X_b are eigenvectors of a normal matrix corresponding to distinct eigenvalues s_a and s_b then $(X_a, X_b) = 0$; that is X_a and X_b are orthogonal.*

The eigenvectors of A corresponding to a multiple eigenvalue can always be orthogonalized by the Gram-Schmidt process. This means that the transformation matrix T, which is composed of the eigenvectors of A can be so chosen as to be unitary. That is $T^*T = I$. We summarize these results in the following:

THEOREM. *If A is normal we can find a unitary matrix T such that TAT^* is diagonal.*

This work was supported by the NSF, under grant GP-165.

A PROOF OF THE PRIME NUMBER THEOREM

E. GROSSWALD, University of Pennsylvania

1. Introduction. Let $\pi(x)$ denote the number of primes not exceeding some real number x and define the symbol of asymptotic equivalence by stipulating that $f(x) \sim g(x)$ shall mean the same as $\lim_{x\to\infty} f(x)/g(x) = 1$. It is our purpose to give a proof of the following statement, known as

THE PRIME NUMBER THEOREM: $\pi(x) \sim x/\log x$.

The proof, while neither as short as Landau's [2], nor as elementary as the proofs of Selberg [3], Erdös [1], or Wright [5], seems to have the advantage of great clarity. Like Landau's proof, it uses only some easily established properties of the Riemann zeta function in the half plane Re $s \geqq 1$.

Let p stand for primes, n for natural integers and define, as usual, $\zeta(s)$ for $s = \sigma + it$ by

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_n n^{-s} \qquad \sigma > 1 \tag{1}$$

and by analytic continuation otherwise. We take for granted the following properties:

(2) $\zeta(s) - (s-1)^{-1}$ and $\dfrac{\zeta'}{\zeta}(s) + (s-1)^{-1}$ are analytic for $\sigma \geqq 1$;

(3) for $s = 1 + it$, $\zeta(s) \neq 0$ and $\zeta(s)$, $\dfrac{\zeta'}{\zeta}(s)$, $\dfrac{\zeta''}{\zeta}(s)$ are all $O(\log^\alpha t)$

$(0 < \alpha < \infty)$ as $t \to \infty$.

Here and in what follows, all logarithms, including $\log \zeta(s)$, are obtained by direct analytic continuation from the branch that is real for real $s > 1$. The first result of (2) (and much more) may be obtained from the Euler-Maclaurin sum formula. The first assertion of (3) follows (after de la Vallée-Poussin) from the observation that (1) leads, for $\sigma > 1$, to $|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geqq 1$, which is inconsistent, on account of (2), with the assumption that $\zeta(1+it) = 0$; the other assertions of (3) may be obtained essentially by the Euler-Maclaurin sum formula. The second assertion of (2) follows from the first, because of (1) and the first of (3). We shall also have to use the following two well-known lemmas:

LEMMA 1 (RIEMANN-LEBESGUE). *Let the function $f(t)$ be differentiable and absolutely integrable on $(0, \infty)$, then the improper integral $J(y) = \int_0^\infty f(t)e^{ity}dt$ converges for every real y and $J(y) = o(1)$ as $y \to \infty$.*

LEMMA 2 (TAUBERIAN). *Let $f(x)$ be positive and nondecreasing; if $\int_1^x u^{-1}f(u)du \sim x(\log x)^{-1}$, then $f(x) \sim x(\log x)^{-1}$.*

2. Sketch of the proof. From (1) with $\sigma > 1$ we obtain

$$\begin{aligned}\log \zeta(s) &= -\sum_p \log(1-p^{-s}) = -\sum_{n=2}^{\infty}\{\pi(n) - \pi(n-1)\}\log(1-n^{-s})\\ &= \sum_{n=2}^{\infty}\pi(n)\{\log(1-(n+1)^{-s}) - \log(1-n^{-s})\}\\ &= \sum_{n=2}^{\infty}\pi(n)\int_n^{n+1}\frac{d}{dx}(\log(1-x^{-s}))dx\\ &= \sum_{n=2}^{\infty}\pi(n)s\int_n^{n+1}x^{-1}(x^s-1)^{-1}dx = s\int_2^\infty x^{-1}(x^s-1)^{-1}\pi(x)dx.\end{aligned}$$

These formal operations are easily justified if $\sigma > 1$. After division by s, the right hand side is almost exactly the Mellin transform of $\pi(x)$. It actually *is* a Mellin transform, not quite of $\pi(x)$, but, as we shall show, of the closely related function $f(x) = \sum_{m=1}^{\infty} m^{-1}\pi(x^{1/m})$. The difference between $f(x)$ and $\pi(x)$ is comparatively small. If q is the greatest integer not exceeding $\log x/\log 2$, then, for $m > q$, $x^{1/m} < 2$ so that $\pi(x^{1/m}) = 0$; hence, $f(x) = \sum_{m=1}^{q} m^{-1}\pi(x^{1/m}) = \pi(x) + \sum_{m=2}^{q} m^{-1}\pi(x^{1/m})$. But

$$0 \leqq \sum_{m=2}^{q} m^{-1}\pi(x^{1/m}) \leqq \sum_{m=2}^{q} m^{-1}x^{1/m} \leqq \sum_{m=2}^{q}\tfrac{1}{2}x^{1/2} = \tfrac{1}{2}(q-1)x^{1/2} < (2\log 2)^{-1}x^{1/2}\log x,$$

so that

$$\pi(x) = f(x) + O(x^{1/2}\log x); \tag{4}$$

clearly, $f(x)=0$ for $x<2$. For later use we also note that $f(x)\leqq \pi(x)+x^{\frac{1}{2}}\log x$, so that, a fortiori,

$$f(x) < 2x. \tag{5}$$

In order to prove the Prime Number Theorem it is sufficient, therefore, to prove that $f(x)=x\ (\log x)^{-1}\cdot(1+o(1))$, because then the same is true of $\pi(x)$, the error term in (4) being of lower order than $o(x(\log x)^{-1})$.

Assuming for a moment that $\int_2^\infty x^{-1}(x^s-1)^{-1}\pi(x)dx=\int_1^\infty f(x)x^{-s-1}dx$, we have obtained so far that, for $\sigma>1$,

$$s^{-1}\log \zeta(s) = \int_1^\infty f(x)x^{-s-1}dx. \tag{6}$$

Equation (6) can be "solved" for $f(x)$, by the classical theorem on the inversion of Mellin transforms, which yields

$$f(x) = (2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-1}x^s\log\zeta(s)ds, \tag{7}$$

valid for any $c>1$. And once we have found $f(x)$, our problem is completely solved by (4), which gives $\pi(x)$.

One may indeed attempt to evaluate the integral in (7) directly. By (2), $\log\ \zeta(s)=-\log\ (s-1)+\log\ h(s)$, with $\log\ h(s)=\log((s-1)\zeta(s))$ analytic in $\sigma\geqq 1$, and, by routine computations (almost identical to those that we shall perform here)

$$-(2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-1}x^s\log\,(s-1)ds$$

is found to be equal to $x(\log x)^{-1}+o(x(\log x)^{-1})$. The proof that the "error term"

$$\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-1}x^s\log h(s)ds$$

is sufficiently small is not trivial. While one may now take even $c=1$ (because $\log h(s)$ stays analytic for $s=1+it$) Lemma 1 is not directly applicable; indeed, for $s=1+it$, (3) only shows that $s^{-1}\log h(s)=O(t^{-1}\log t)$. This is not sufficient to insure the absolute convergence of the integral, which would be the simplest way to show that the convergence is uniform with respect to x.

In order to avoid these difficulties, it is preferable to return for a moment to (6) for a very slight change which will insure the absolute convergence of the integrals involved. For that purpose, set $g(x)=\int_1^x u^{-1}f(u)du$; then $g(x)=0$ for $x<2$ (because $f(x)=0$ for $x<2$) and, by (5), $g(x)<\int_1^x u^{-1}(2u)du<2x$. Also, $g'(x)=x^{-1}f(x)$ and, after an integration by parts, the second member of (6) may be rewritten as

$$\int_1^\infty g'(x)x^{-s}dx = g(x)x^{-s}\Big|_1^\infty + s\int_1^\infty g(x)x^{-s-1}dx.$$

For $\sigma > 1$, the integrated term vanishes and instead of (6) we obtain

$$(6')\qquad s^{-2}\log\zeta(s) = \int_1^\infty g(x)x^{-s-1}dx \qquad (\sigma > 1);$$

hence instead of (7) we obtain

$$(7')\qquad g(x) = (2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-2}x^s\log\zeta(s)ds \qquad (c > 1).$$

If we replace $\log\zeta(s)$, as before, by $-\log(s-1)+\log h(s)$, then (7′) becomes

$$(8)\qquad g(x) = I_1(x) + I_2(x),$$

where

$$I_1(x) = -(2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-2}x^s\log(s-1)ds,$$

$$I_2(x) = (2\pi i)^{-1}\lim_{T\to\infty}\int_{c-iT}^{c+iT} s^{-2}x^s\log h(s)ds.$$

We shall show first that $I_2(x)=o(x(\log x)^{-1})$; next, that $I_1(x)=x(\log x)^{-1}+o(x(\log x)^{-1})$. Then it follows from (8) that $g(x)=x(\log x)^{-1}+o(x(\log x)^{-1})$. Lemma 2 leads then to $f(x)=x(\log x)^{-1}+o(x(\log x)^{-1})$ and, on account of (4), the theorem will be proven.

3. Proof of the lemmas. For completeness, we indicate here the proof of the lemmas used.

Proof of Lemma 1. The function $f(t)$ being absolutely integrable, we can determine T_1 and ϵ_1 so that, for $T_1 \leqq T$, $\epsilon \leqq \epsilon_1$

$$\left|\int_T^\infty f(t)e^{ity}dt\right| \leqq \int_T^\infty |f(t)|\,dt < \tfrac{1}{3}\eta, \left|\int_0^\epsilon f(t)e^{ity}dt\right| \leqq \int_0^\epsilon |f(t)|\,dt < \frac{\eta}{3}$$

for arbitrarily small $\eta > 0$. Next, keeping T and ϵ fixed and integrating by parts, we obtain

$$\left|\int_\epsilon^T f(t)e^{ity}dt\right| = \left|(iy)^{-1}\left\{e^{ity}f(t)\Big|_\epsilon^T - \int_\epsilon^T f'(t)e^{ity}dt\right\}\right|$$

$$\leqq y^{-1}\left\{|f(T)| + |f(\epsilon)| + \int_\epsilon^T |f'(t)|\,dt\right\} < \tfrac{1}{3}\eta$$

for sufficiently large y and the lemma follows.

Proof of Lemma 2. Given any $\epsilon > 0$, set $y = x(1+\epsilon)$ and, for sufficiently large x, consider the quantity $\phi = \int_x^y u^{-1}f(u)du$. On the one hand, by hypothesis,

$$(1 - \epsilon^2)x(\log x)^{-1} < \int_1^x u^{-1}f(u)du < (1 + \epsilon^2)x(\log x)^{-1},$$

so that

$$\begin{aligned}\phi = \int_1^y u^{-1}f(u)du - \int_1^x u^{-1}f(u)du &< (1 + \epsilon^2)y(\log y)^{-1} - (1 - \epsilon^2)x(\log x)^{-1} \\ &< (\log x)^{-1}\{(1 + \epsilon^2)y - (1 - \epsilon^2)x\} = x(\log x)^{-1}\{(1 + \epsilon^2)(1 + \epsilon) - (1 - \epsilon^2)\} \\ &= x(\log x)^{-1}\epsilon(1 + \epsilon)^2;\end{aligned}$$

on the other hand, by the monotonicity of $f(x)$, $\phi = \int_x^y u^{-1}f(u)du \geqq f(x)\int_x^y u^{-1}du = f(x)\log(y/x) = f(x)\log(1+\epsilon)$. Hence,

$$f(x) \leqq \phi/\log(1 + \epsilon) < x(\log x)^{-1}\{\epsilon(1 + \epsilon)^2/\log(1 + \epsilon)\} \leqq x(\log x)^{-1}(1 + \epsilon)^3$$

and $(x^{-1}\log x)f(x) < (1+\epsilon)^3$ for arbitrarily small $\epsilon > 0$; similarly, one shows that $(x^{-1}\log x)f(x) > (1-\epsilon)^3$ for arbitrarily small $\epsilon > 0$, if only x is large enough and this finishes the proof of the Lemma.

4. Proof of the theorem. It only remains to fill in the details of the different steps sketched without proofs in Section 2.

(a) *Proof of* (6). From $f(x) = \sum_{m=1}^{\infty} m^{-1}\pi(x^{1/m}) = \sum_{m=1}^{q} m^{-1}\pi(x^{1/m})$ it follows that

$$\int_1^\infty f(x)x^{-s-1}dx = \int_1^\infty \sum_{m=1}^{q} m^{-1}\pi(x^{1/m})\cdot x^{-s-1}dx = \sum_{m=1}^{\infty}\int_1^\infty m^{-1}\pi(x^{1/m})x^{-s-1}dx,$$

because of the uniform convergence of the series. In each integral we make the change of variable $x = y^m$, obtaining

$$\begin{aligned}\int_1^\infty f(x)x^{-s-1}dx = \sum_{m=1}^{\infty}\int_1^\infty \pi(y)y^{-ms-1}dy &= \int_1^\infty\left(\sum_{m=1}^{\infty}\pi(x)x^{-ms-1}\right)dx \\ &= \int_1^\infty x^{-1}(x^s - 1)^{-1}\pi(x)dx,\end{aligned}$$

the termwise integration being again justified by the uniform convergence of the series (for $1 \leqq x < \infty$, and constant $\sigma > 1$).

(b) *Estimation of* $I_2(x)$. By Cauchy's theorem on residues,

$$I_2(x) = (2\pi)^{-1}\lim_{T\to\infty}\int_{-T}^{+T}(1 + it)^{-2}x^{1+it}g_1(t)dt,$$

where $g_1(t) = \log h(1+it)$. Hence, setting

$$I_3(y) = \lim_{T\to\infty} \int_{-T}^{+T} (1+it)^{-2} g_1(t) e^{ity} dt, \qquad I_2(x) = (2\pi)^{-1} x I_3(\log x).$$

Integrating by parts, we obtain

$$I_3(y) = \lim_{T\to\infty} \left\{ (iy)^{-1} e^{ity}(1+it)^{-2} g_1(t) \Big|_{-T}^{+T} - (iy)^{-1} \int_{-T}^{+T} (1+it)^{-3} g_2(t) e^{ity} dt \right\}$$

with

$$\begin{aligned} g_2(t) &= (1+it) g_1'(t) - 2i g_1(t) \\ &= i\left\{ s\left((s-1)^{-1} + \frac{\zeta'}{\zeta}(s) \right) - 2\log((s-1)\zeta(s)) \right\}_{s=1+it}. \end{aligned}$$

Using (3), we see that $g_1(t)$ and $g_2(t)$ are both differentiable for real t,

$$g_1(t) = \log\{(s-1)\zeta(s)\}_{s=1+it} = \log t + O(\log\log t),$$

and $g_2(t) = o(t \log^\alpha t)$ for $t\to\infty$. Hence, the integrated term of $I_3(y)\to 0$ as $T\to\infty$ and Lemma 2 is applicable to the last integral. It follows, as claimed, that

$$I_3(y) = o(y^{-1}) \quad \text{and} \quad I_2(x) = (2\pi)^{-1} x I_3(\log x) = o(x(\log x)^{-1}).$$

(c) *Computation of* $I_1(x)$. In $I_1(x)$ we move the line of integration to $c=1$, with a small semi-circular indentation Γ around the singularity $s=1$. This is permitted by Cauchy's theorem on residues, because the integrand goes to zero as $t\to\infty$ and has no singularities for $\sigma \geqq 1$, $s\neq 1$. Hence,

$$-2\pi i I_1(x) = \lim_{T\to\infty} \left\{ \int_{1-iT}^{1-i\eta} + \int_\Gamma + \int_{1+i\eta}^{1+iT} s^{-2} x^s \log(s-1) ds \right\}.$$

The contribution of the integral along Γ can be made arbitrarily small, by taking η sufficiently small. Indeed,

$$\begin{aligned} \left| \int_\Gamma s^{-2} x^s \log(s-1) ds \right| &= \left| \int_{-\pi/2}^{\pi/2} (1+\eta e^{i\theta})^{-2} x^{1+\eta e^{i\theta}} \log(\eta e^{i\theta}) \cdot \eta i e^{i\theta} d\theta \right| \\ &\leqq (1-\eta)^{-2} x^{1+\eta} \cdot \eta \int_{-\pi/2}^{\pi/2} |\log(\eta e^{i\theta})| d\theta \\ &\leqq (1-\eta)^{-2} x^{1+\eta} \eta \int_{-\pi/2}^{\pi/2} (\log \eta^{-1} + |\theta|) d\theta \\ &= (1-\eta)^2 x^{1+\eta} \eta \{\pi \log \eta^{-1} + \tfrac{1}{4}\pi^2\} \to 0 \end{aligned}$$

as $\eta\to 0$. Hence,

$$\lim_{T\to\infty} \int_{c-iT}^{c+iT} s^{-2} x^s \log(s-1) ds = ix \lim_{\eta\to 0} \lim_{T\to\infty} \left\{ \int_{-T}^{-\eta} + \int_\eta^T (1+it)^{-2} x^{it} \log(it) dt \right\},$$

and $I_1(x)$ becomes $-(2\pi)^{-1} x I(\log x)$, with

$$I(y) = \lim_{\eta\to 0} \lim_{T\to\infty} \left\{ \int_{-T}^{-\eta} + \int_{\eta}^{T} (1 + it)^{-2} \log (it) e^{ityd t} \right\}.$$

We may shorten the computations slightly by observing that the integral $\int_{-T}^{-\eta} \cdots dt$ is the complex conjugate of the integral $\int_{\eta}^{T} \cdots dt$. The latter one is computed explicitly. By an integration by parts,

$$\int_{\eta}^{T} (1 + it)^{-2}(\log t + \tfrac{1}{2}\pi i)e^{ity}dt = (iy)^{-1}e^{ity}(1 + it)^{-2}(\log t + \tfrac{1}{2}\pi i)\Big|_{\eta}^{T}$$
$$- (iy)^{-1} \int_{\eta}^{T} (1 + it)^{-3}\{t^{-1} + i + \pi - 2i \log t\} e^{ity}dt.$$

Here the integral $\int_{\eta}^{T}(1+it)^{-3}(i+\pi-2i \log t)e^{ity}dt$ satisfies the conditions of Lemma 1, so that it is $o(1)$. If we let $T\to\infty$, the integrated term vanishes at the upper limit and we remain with

$$\lim_{T\to\infty} \int_{\eta}^{T} (1 + it)^{-2} \log (it)e^{ity}dt$$
$$= - (iy)^{-1}\left\{(1 + i\eta)^{-2}(\log \eta + \tfrac{1}{2}\pi i)e^{i\eta} + \int_{\eta}^{\infty} t^{-1}(1 + it)^{-3}e^{ity}dt\right\}.$$

Adding the complex conjugate and taking the limit as $\eta\to 0$, we obtain

$$I(y) = - (iy)^{-1} \lim_{\eta\to 0} \left\{\frac{\log \eta + \frac{1}{2}\pi i}{(1 + i\eta)^2} e^{iy\eta} - \frac{\log \eta - \frac{1}{2}\pi i}{(1 - i\eta)^2} e^{-iy\eta}\right.$$
$$\left. + \int_{\eta}^{\infty} \left\{\frac{e^{ity}}{t(1 + it)^3} - \frac{e^{-ity}}{t(1 - it)^3}\right\} dt + o(y^{-1})\right.$$
$$= - (iy)^{-1} \lim_{\eta\to 0} \left\{[\log \eta\cdot(e^{iy\eta} - e^{-iy\eta}) + \tfrac{1}{2}\pi i(e^{iy\eta} + e^{-iy\eta})](1 + O(\eta))\right.$$
$$\left. + \int_{\eta}^{\infty} \frac{e^{ity} - e^{-ity}}{t} dt + \int_{\eta}^{\infty} \frac{p_1(t)e^{ity} + p_2(t)e^{-ity}}{(1 + t^2)^3} dt\right\} + o(y^{-1}).$$

Here $p_1(t)$ and $p_2(t)$ are fifth degree polynomials in t; hence, by an integration by parts and Lemma 1, the last integral is $O(\eta y^{-1})+o(y^{-1})$ and we obtain

$$I(y)=-y^{-1} \lim_{\eta\to 0} \left\{(2 \log \eta\cdot \sin y\eta+\pi\cos y\eta)(1+O(\eta))+2\int_{\eta}^{\infty} \frac{e^{ity}-e^{-ity}}{2it} dt\right\} +o(y^{-1})$$
$$=-\pi y^{-1}-2y^{-1}\int_{0}^{\infty} t^{-1} \sin ty\, dt+o(y^{-1})=-y^{-1}\left(\pi+2\int_{0}^{\infty} v^{-1} \sin v\, dv\right)$$
$$+o(y^{-1})=-2\pi y^{-1}+o(y^{-1}).$$

Hence, $I_1(x)=-(2\pi)^{-1}xI(\log x)$ becomes $I_1(x)=x(\log x)^{-1}+o(x(\log x)^{-1})$ and, by (8), $g(x)=x(\log x)^{-1})+o(x(\log x)^{-1})$. As we have already seen, it now follows from Lemma 2 that $f(x)=x(\log x)^{-1}+o(x(\log x)^{-1})$ and, on account of (4) the proof is complete.

This paper was written with the support of the National Science Foundation Contract NSF-G 24348.

References

1. P. Erdös, Proc. Nat. Acad. Sci., 35 (1949) 374–384.
2. E. Landau, Sitzungsberichte, Preuss. Akad. Wiss., (1932) 514–521.
3. A. Selberg, Ann. of Math., 50 (1949) 305–313.
4. E. C. Titchmarsh, The Theory of the Riemann Zeta Function, Clarendon Press, Oxford, 1951.
5. E. M. Wright, Proc. Roy. Soc. Edinburgh, 6 (1952) 257–267.

ON GENERATORS AND DEFINING RELATIONS FOR THE UNIMODULAR GROUP $\mathfrak{M}_2$

GEORGE K. WHITE, University of Toronto

1. Introduction. In this paper, $\mathfrak{M}_n$ $(n=2, 3, \cdots)$ shall denote the unimodular group of all real $n\times n$ matrices with integral entries and determinant ± 1.

Burrowes Hunt [5; 6] has for some time interested himself and his student Beldin in the unimodular groups, and the latter [1] has given in his thesis (unpublished) an expression which yields for any given n two elements $A, B\in\mathfrak{M}_n$ which suffice to generate $\mathfrak{M}_n$. Trott [7] has since deduced independently another pair of generators U_2, U for $\mathfrak{M}_n$. Two other recent papers of interest, kindly brought to my attention by Professor Hunt, are one by Brenner [2] (who finds yet another pair of generators S, T' of $\mathfrak{M}_n$), and one by Sze-Chien Yien [8] who gives a set $\{B_{ij}, U\}$ of more than two generators, and goes on to find complete defining relations for all $\mathfrak{M}_n$, using the sets $\{B_{ij}, U\}$.

This brief bibliography by no means exhausts the literature: there are, for instance, further references in [3], chapter 7. In this paper we restrict our attention to the particular group $\mathfrak{M}_2$, and the Beldin generators $\{A, B\}$. Our goal is to supply a simple abstract 2-generator definition of $\mathfrak{M}_2$ by providing suitable defining relations for A, B. We achieve this in sections 2 and 3 by first imposing relations (3) on the symbols A, B to define an abstract group $\mathfrak{H}$, and then proving that $\mathfrak{H}\simeq\mathfrak{M}_2$ (Lemma 1) utilizing results set forth in [3]. In section 4, we make use of the Fibonacci numbers to deduce and state some further simple relations between A, B, and to find concise expressions for $Z=-E$ (where E denotes the identity), in terms of A, B.

I should like to thank Professor Coxeter for his interest and help in connection with this paper. I am also grateful to Professor Hunt for, among other things, providing us with material from Beldin's thesis.