3042. On Mersenne's Primes, Fermat's Primes and Even Perfect Numbers
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Theorem I. Let n be an odd integer >1. A necessary and sufficient condition for n to be a Mersenne's prime is that

$$p^2-1\equiv 0 \pmod{\tfrac{1}{2}n(n+1)},$$

p being the product of all positive odd integers $<n$.

Theorem II. Let n be an even integer >2. A necessary and sufficient condition for $(n+1)$ to be a Fermat's prime is that

$$(p-1)(p^2+1)\equiv 0 \pmod{\tfrac{1}{2}n(n+1)},$$

p being the product of all positive odd integers $<n$.

Theorem III. For any integer $x>3$ to be an even perfect number, it is necessary and sufficient that (i) x is a triangular number and (ii) $p^2-1\equiv 0 \pmod{x}$, p being the product of all positive odd integers $<n$, where $x=\frac{1}{2}n(n+1)$ (the nth triangular number).

Proof of Theorem I. Suppose n is a Mersenne's prime, say $2^{r+1}-1$.

$$p=1\,.\,3\,.\,5\,.\,\ldots\,.\,(2^{r+1}-3)\equiv -(2^{r+1}-2)\,.\,-(2^{r+1}-4)\,.\,\ldots\,.\,-2 \pmod{2^{r+1}-1}$$

$$\therefore\ p^2\equiv -(2^{r+1}-2)! \pmod{2^{r+1}-1}$$

$$\equiv 1 \pmod{2^{r+1}-1}, \text{ by Wilson's theorem.} \qquad (1)$$

Now consider the two sets of numbers, viz; $1, 3, 5, \ldots, 2^r-1$ and $2^r+1, 2^r+3, \ldots, 2^{r+1}-3, 2^{r+1}-1$. Each set is clearly a reduced residue system modulo 2^r. Therefore to each member l of the first set corresponds a unique member l' of the second set such that $ll'\equiv 1 \pmod{2^r}$. Taking the product of all such congruences (2^{r-1} congruences in number), we have

$$p(2^{r+1}-1)\equiv 1 \pmod{2^r}$$

$$\therefore\ p^2\equiv 1 \pmod{2^r} \qquad (2)$$

Since $(2^r, 2^{r+1}-1)=1$, from (1) and (2) we see that

$$p^2-1\equiv 0 \pmod{\tfrac{1}{2}n(n+1)}.$$

Thus the condition is necessary.

If $p^2-1\equiv 0 \pmod{n(n+1)/2}$, it immediately follows that n is a prime and $n+1$ is a power of 2, since n is odd >1; so that n is a Mersenne's prime. Hence the condition is sufficient.

Proof of Theorem II. Suppose $(n+1)$ is a Fermat's prime, say $2^{2r}+1$. Since n is an even integer >2, $r\geqslant 1$.

$$p=1\,.\,3\,.\,5\,.\,\ldots\,.\,(2^{2r}-1)\equiv -2^{2r}\,.\,-(2^{2r}-2)\,.\,\ldots\,.\,-2 \pmod{2^{2r}+1}$$

Since $r\geqslant 1$, $p^2\equiv (2^{2r})! \pmod{2^{2r}+1}$

$$\equiv -1 \pmod{2^{2r}+1} \text{ by Wilson's theorem.}$$

Using the same argument as in the proof of theorem I, we can prove that $p \equiv 1 \pmod{2^{2r-1}}$.

Thus we see that

$$(p-1)(p^2+1) \equiv 0 \pmod{\tfrac{1}{2}n(n+1)}$$

and so the condition is necessary.

If $(p-1)(p^2+1) \equiv 0 \pmod{\tfrac{1}{2}n(n+1)}$, it immediately follows that $(n+1)$ is a prime and $\tfrac{1}{2}n$ is a power of 2, since n is even >2. By the well-known result that if 2^t+1 is a prime, then t is a power of 2, it follows that $(n+1)$ is a Fermat's prime. Hence the condition is sufficient.

Proof of Theorem III. Suppose x is an even perfect number. It is well known that x must be of Euclid's type, namely $2^r(2^{r+1}-1)$ where $2^{r+1}-1$ is a prime. Clearly x is the $(2^{r+1}-1)$th triangular number, so that the condition (i) is satisfied and (ii) is also satisfied in virtue of Theorem I. Thus the conditions are necessary.

If $x > 3$ is a triangular number, say $\tfrac{1}{2}n(n+1)$ and $p^2-1 \equiv 0 \pmod{x}$, p being the product of all positive odd integers $<n$, we shall prove that n is odd >1. Since $x>3$, $n>2$. Suppose if possible that n is even; then by the above congruence it is clear that $\tfrac{1}{2}n$ is a power of 2, say 2^r and $n+1 = 2^{r+1}+1$ is a prime. By the result already mentioned in the proof of theorem II, it follows that $(n+1)$ is a Fermat's prime and hence $p^2+1 \equiv 0 \pmod{(n+1)}$. But by hypothesis we have $p^2-1 \equiv 0 \pmod{(n+1)}$. Therefore $(n+1)$ must divide 2 which is a contradiction to the supposition that n is even. Thus n is odd >1. By theorem I, it follows that n is a Mersenne's prime; so that x is an even perfect number. Hence the conditions are sufficient.

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3043. Volume of a triangular pyramid

Let O, P_1, P_2, P_3 be the vertices and V the volume. The following two determinantal formulae

$$36V^2 = OP_1^2 OP_2^2 OP_3^2 \begin{vmatrix} 1 & \cos(1,2) & \cos(1,3) \\ \cos(2,1) & 1 & \cos(2,3) \\ \cos(3,1) & \cos(3,2) & 1 \end{vmatrix} \qquad (1)$$

and

$$36V^2 = \begin{vmatrix} OP_1^2 & (OP_1^2+OP_2^2-P_1P_2^2)/2 & (OP_1^2+OP_3^2-P_1P_3^2)/2 \\ (OP_2^2+OP_1^2-P_2P_1^2)/2 & OP_2^2 & (OP_2^2+OP_3^2-P_1P_3^2)/2 \\ (OP_3^2+OP_1^2-P_3P_1^2)/2 & (OP_3^2+OP_2^2-P_3P_2^2)/2 & OP_3^2 \end{vmatrix} \qquad (2)$$