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ADDITIVE PRIME NUMBER THEORY*

By L. Mirsky

The scope of the additive prime number theory is evident from its name. In this part of the theory of numbers we are concerned with the representation of integers as sums of primes; and here the central problem consists in the proof (or possibly refutation) of a celebrated conjecture made by Goldbach in 1742 to the effect that every even integer $\geqslant 4$ can be represented as the sum of 2 primes. The truth of this conjecture is still unsettled; but though progress in this field has been slow, attempts to deal with the problem have led to a whole series of striking results.

Goldbach's problem remained virtually untouched till 1923 when Hardy and Littlewood attacked it by means of a new and powerful analytical technique, which has come to be known (from a contour integral employed in the argument) as the "circle method." The work of Hardy and Littlewood showed that the problem was tractable; but the success of their method was incomplete, for they were only able to establish their results on the assumption of a certain hypothesis in the theory of functions which remains unproved to the present day.†

By combining the ideas of Hardy and Littlewood with daringly ingenious inequalities for exponential sums, Vinogradoff succeeded in 1937 in dispensing with H and putting on a firm foundation all earlier hypothetical statements. The most important result proved in this way was as follows.

Theorem 1. *Every sufficiently large odd integer can be represented as the sum of* 3 *primes.*

This result follows, of course, trivially from Goldbach's conjecture, but the converse inference cannot be made.

Vinogradoff proved, in fact, much more than Theorem 1. Let $r_3(n)$ denote the number of representations of n as the sum of 3 primes. Vinogradoff showed that, as $n \to \infty$ through odd values,‡

$$r_3(n) \sim \frac{n^2}{2\log^3 n} \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|n} \left(1 - \frac{1}{p^2 - 3p + 3}\right). \qquad \ldots\ldots(1)$$

An immediate corollary of Theorem 1 may also be noted.

Theorem 2. *Every sufficiently large even integer can be represented as the sum of* 4 *primes.*

Another interesting result which emerged from Vinogradoff's work is as follows. Let $N(x)$ denote the number of even integers $\leqslant x$ which are *not* representable as the sum of 2 primes; then $N(x)/x \to 0$ as $x \to \infty$. Thus Goldbach's conjecture is true at any rate in the majority of cases, and this assertion is usually expressed in the following form.

Theorem 3. *Almost all even integers can be represented as sums of* 2 *primes.*

All three theorems stated above, as well as the asymptotic formula (1), were first established by Hardy and Littlewood on the basis of the unproved hypothesis H.

* Shortened version of an address given to the British Mathematical Colloquium at St. Andrews in September, 1956.

† This hypothesis, which we shall refer to as H, is a generalization of the famous "Riemann hypothesis."

‡ The symbol $\sim$ in (1) indicates that the ratio of the two sides tends to unity. The letter p is reserved for primes throughout; and in the first product on the right-hand side of (1) p ranges over all primes, while in the second it ranges over all prime divisors of n.

The development of the circle method at the hands of Hardy, Littlewood, and Vinogradoff is probably comparatively well known, and we therefore propose to consider in greater detail an alternative treatment of problems in the additive theory of numbers. This is the "sieve method," which was devised originally by Viggo Brun in 1915–20 and has since been modified and extended by other writers. The sieve method is elementary and does not lead to the complete solution of any problem but, compared with the circle method, it has the advantage of being applicable to a wider range of questions.

Consider the sequence of integers 1, 2, ..., n and let $p_1, \ldots, p_r$ be any r distinct primes. From this sequence we delete all integers which belong to at least one of r specified arithmetic progressions with common differences $p_1, \ldots, p_r$ respectively, i.e. we delete one residue class (mod p_1), one residue class (mod p_2), and so on. More generally, it may be necessary to delete $m \geqslant 1$ residue classes modulo each of the primes $p_1, \ldots, p_r$, and this procedure is described as "sieving." Brun gave a method for estimating the number of integers in the sequence 1, 2, ..., n which survive the process of sieving, and he showed how such estimates could be used for dealing with questions which had previously seemed intractable.

An adaptation of Brun's sieve method enabled Schnirelmann, in 1930, to make a very substantial advance in the direction of Goldbach's conjecture. Denote by $p_1, \ldots, p_k$ all the primes $\leqslant \sqrt{n}$, and by $Q(n)$ the number of positive integers $t \leqslant n$ such that

$$t \not\equiv 0, \qquad t \not\equiv n \pmod{p_i} \qquad (i = 1, \ldots, k).$$

Thus $Q(n)$ is the number of integers $\leqslant n$ which do not belong to two specified residue classes modulo each of the primes $p_1, \ldots, p_k$. Using the sieve method, Schnirelmann showed that*

$$Q(n) < c_1 \frac{n}{\log^2 n} \sum_{d|n} d^{-1}, \qquad \ldots\ldots(2)$$

where in the summation d ranges over all divisors of n. Next, let $r_2(n)$ denote the number of representations of n as the sum of two primes, i.e. the number of solutions of the equation

$$n = p' + p''. \qquad \ldots\ldots(3)$$

The number of solutions of (3), subject to $p' > \sqrt{n}$, $p'' > \sqrt{n}$, does not exceed $Q(n)$; while the number of solutions subject to $\min(p', p'') \leqslant \sqrt{n}$ does not exceed $2\sqrt{n}$. Hence

$$r_2(n) \leqslant Q(n) + 2\sqrt{n},$$

and so, by (2),

$$r_2(n) < c_2 \frac{n}{\log^2 n} \sum_{d|n} d^{-1}.$$

Thus $r_2(n)$ is never very large. On the other hand, the average value of $r_2(n)$ is not too small, for

$$\sum_{n \leqslant x} r_2(n) = \sum_{p'+p'' \leqslant x} 1 > c_3 \frac{x^2}{\log^2 x}.$$

Now if a non-negative function is not too small on the average and not too large for any value of the variable n, then it must be positive for many values of n. The precise formulation of this idea enables us to infer that the number $E(n)$ of integers $t \leqslant n$ such that $r_2(t) > 0$ (i.e. the number of integers $\leqslant n$ representable in the sense of Goldbach) satisfies the inequality $E(n) > c_4 n$.

* The letter c is reserved for absolute positive constants.

In other words, the sequence of "representable" integers has positive density.* Now it was shown by Schnirelmann—and the proof is not at all difficult—that if a sequence $\{s_i\}$ of integers has positive density δ, then there exists a number $l = l(\delta)$ such that every integer can be represented as the sum of at most l elements of $\{s_i\}$. It follows, therefore, that every integer is the sum of at most c_5 "representable" integers, and so the sum of at most $2c_5 = c_6$ primes.

The value of c_6 which emerged from Schnirelmann's analysis was large, namely 800,000. Nevertheless, the theorem that every integer is the sum of at most 800,000 primes constituted a very large step forward, since at the time only the hypothetical results of Hardy and Littlewood were available. Subsequent writers improved on the quantitative aspect of Schnirelmann's work. It was shown, eventually, that every sufficiently large number is the sum of at most 67 primes, but even this result was soon superseded by Vinogradoff's Theorems I and II.

So far we have been discussing the representation of integers as sums of more than 2 primes, but a relaxation of Goldbach's conjecture can clearly take different forms and we might, for example, consider the representation of n in the form $n = n_1 + n_2$, where the conditions imposed on n_1 and n_2 are less severe than those of primality. Brun employed his sieve method in an investigation of this type and showed that every sufficiently large integer can be represented as the sum of 2 integers each of which has at most 9 prime divisors. This result was sharpened in successive stages till, in 1940, Buchstab proved

THEOREM 4. *Every sufficiently large integer can be represented as the sum of* 2 *integers each of which has at most* 4 *prime divisors.*†

A variant of the problem just considered consists in investigating the existence of representations of n in the form $n = p + n'$, where n' has few prime divisors. Already in 1932 Estermann had shown that, subject to a hypothesis slightly stronger than H, every sufficiently large integer n possesses such a representation, with n' having at most 6 prime divisors. In 1948 Renyi proved an extraordinarily interesting theorem which, though weaker than Estermann's, does not depend on any unproved result. Renyi's theorem is as follows.

THEOREM 5. *Every integer can be represented as the sum of a prime and a number having at most c_7 prime divisors.*

The proof of this result is long and difficult and makes full use both of arithmetical and analytical techniques. Probably the most interesting feature of the proof is a powerful generalization of Brun's sieve method, and we shall conclude by briefly referring to this aspect of Renyi's work. It will be recalled that the sieve method, as devised by Brun, is concerned with estimating the number h of integers $\leqslant n$ which do not belong to m specified residue classes modulo each of the primes $p_1, \ldots, p_r$. Consider now the more general case when the number of excluded residue classes (mod p_i), instead of being a fixed number m, is some function $f(p_i)$ of p_i. Brun's method for estimating h is now no longer applicable, but in 1941 Linnik invented a technique for dealing with this new operation—an operation which he aptly termed 'the large sieve.' Linnik's theorem states that *if $p_1, \ldots, p_r$ are any primes $\leqslant \sqrt{n}$ and $\sigma = \min f(p_i)/p_i$, then $h \leqslant c_8 n/\sigma^2 r$.* Even this result was insufficient for Renyi's purpose and a yet further generalization of the sieve method was needed before Theorem 5 could be established.

There are, of course, results bearing on Goldbach's conjecture other than those we have mentioned. Thus, for example, a rather amusing theorem due to

* The density of the sequence $\{s_i\}$ is defined as the lower bound of $S(n)/n$, where $S(n)$ denotes the number of s_i which do not exceed n.

† A. Selberg stated in 1949 that it is possible to replace the number 4 in Theorem 4 by 3, but he has not published a detailed proof.

Linnik (1953) states that if any even integer is written in the binary scale, then it is possible to convert it into a "Goldbach number" by changing at most c_9 digits. However, the five theorems listed above are certainly the most important known approximations to Goldbach's conjecture. In a sense they all come near to the desired end, yet the final proof still eludes us; and it would probably be rash to hazard a guess whether we are as yet anywhere near the definitive solution of the problem.

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1900. But, my dear Mrs. Witham, indeed you need not be concerned about me! A man who is reading for the Mathematical Tripos has too much to think of to be disturbed by any of these mysterious "somethings," and his work is of too exact and prosaic a kind to allow of his having any corner in his mind for mysteries of any kind. Harmonical Progressions, Permutations and Combinations, and Elliptic Functions have sufficient mysteries for me!—Bram Stoker, *The Judge's House.* [Per Mr. B. Cook.]

1901. It is laid down that the smallest offence against God being committed in the sight of an infinitely wise and good being, "who is of purer eyes than to behold iniquity," must appear like a spot of the deepest dye, must cause an infinite degree of disapprobation, and subject the offender to an unmitigated and never-ending punishment in consequence.... Can anyone suppose that Sir Isaac Newton, with the accessions of knowledge which his mild spirit may have acquired in the starry sphere, would be thrown into convulsions of rage and agony on beholding some mistake which a poor village schoolmaster had committed in a sum in arithmetic.—William Hazlitt, *Complete Works* (Centenary Edition) XX, p. 225. [Per Professor T. A. A. Broadbent.]

1902. In order to prove that a guinea and a feather would descend *in vacuo* in the same time, he (Isaac Milner, Lucasian Professor at Cambridge, 1798–1820) made use of a glass tube hermetically sealed in, in which the guinea and the feather were enclosed; it so happened, that in several attempts the guinea had the advantage; he then managed to place the guinea above the feather. At the end he exclaimed, "How beautifully this experiment has succeeded! For if you observed attentively, you would perceive that the feather was down sooner than the guinea".—H. Gunning, *Reminiscences of the University, Town and Country of Cambridge*, (2nd edition, 1855), Vol. I, p. 236. [Per Professor T. A. A. Broadbent.]

1903. Accounts under the Metric System. Accordingly the gross was adopted as the basic unit with prices quoted in shillings and decimals of a shilling. A single article becomes 0·007 of a gross, a penny is reckoned as 0·083 of a shilling, and only the total price of a quotation is given in £ s. d. Thus the customer now receives an invoice which instead of quoting say three gross, seven dozen articles at £2 10s. 6d. a gross, refers to 3·583 gross at 50·5 shillings a gross

So far only once customer has complained. On receiving his first invoice he telephoned to say he could not understand it. For his special benefit quantities and prices in the old reckoning are now inserted parenthetically on invoices sent to his firm.—*The Times*, 14th March 1957. [Per Mr. W. H. Cozens.]

1904. Question and Answer. Can a sailing boat sail faster than the wind?—*Arthur Pells, Cowbridge road, Cardiff.*

Not if the wind is astern, but it can sail much faster than the wind if this is at right angles to the boat's course.—*Daily Herald*, 25th April 1957. [Per Mr. W. H. Cozens.]