MAT 211 Introduction to Linear Algebra
Spring 2011
Final exam

Problem 1. Find all solutions of the system

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & a \\ x & + & 3y & + & 8z & = & b \\ x & + & 2y & + & 2z & = & c \end{array}\right|,$$

where a, b and c are arbitrary constants.

Solution. We use Gauss–Jordan elimination:

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & a \\ x & + & 3y & + & 8z & = & b \\ x & + & 2y & + & 2z & = & c \end{array}\right| \begin{array}{l} \\ -1 \cdot I \rightarrow \\ -1 \cdot I \end{array} \left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & a \\ & & y & + & 5z & = & b-a \\ & & & & -z & = & c-a \end{array}\right| \begin{array}{l} -2 \cdot II \\ \qquad\qquad \rightarrow \\ \end{array}$$

$$\left|\begin{array}{ccccc} x & & - & 7z & = & 3a-2b \\ & y & + & 5z & = & b-a \\ & & & -z & = & c-a \end{array}\right| \begin{array}{l} \\ \qquad \rightarrow \\ \times(-1) \end{array} \left|\begin{array}{ccccc} x & & - & 7z & = & 3a-2b \\ & y & + & 5z & = & b-a \\ & & & z & = & a-c \end{array}\right| \begin{array}{l} +7 \cdot III \\ -5 \cdot III \rightarrow \\ \end{array}$$

$$\left|\begin{array}{ccccc} x & & & = & 10a-2b-7c \\ & y & & = & -6a+b+5c \\ & & z & = & a-c \end{array}\right|$$

Problem 2. Let T be the linear transformation with matrix

$$A = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

(with respect to the standard basis).

1. Find a vector $\vec{v}_1$ that spans the kernel of T and a vector $\vec{v}_2$ that spans the image of T.
2. Let $\mathcal{B}$ be the basis consisting of $\vec{v}_1$ and $\vec{v}_2$. Find the matrix B of T with respect to the basis $\mathcal{B}$.
3. Describe the transformation T geometrically.

Solution. We see that the second column vector of A is (-1) times the first, so the vector

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is in the kernel of A (the kernel cannot be two-dimensional, since then A would be the zero matrix). Either of the column vectors spans the image, but it is convenient to rescale them and get rid of the denominators, so we choose

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The matrix B is found using the change of basis formula:

$$B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} == -\frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

In other words, $T(\vec{v}_1) = \vec{0}$ and $T(\vec{v}_2) = \vec{v}_2$. Since $\vec{v}_1$ and $\vec{v}_2$ are orthogonal, we conclude that T is an orthogonal projection onto the line spanned by $\vec{v}_2$, which is the line $x + y = 0$.

Problem 3. Let A be the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

1. Find the reduced row-echelon form of A.
2. Find a basis for the kernel of A.
3. Find a basis for the image of A.

Solution.

First we row-reduce:

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix} \begin{matrix} \\ \\ -3 \cdot I \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 4 & -12 & -4 \\ 0 & -1 & 3 & 1 \end{bmatrix} \begin{matrix} \\ \\ -4 \cdot II \\ +1 \cdot II \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find a basis for the kernel, we look at the free variables. Here x_3 and x_4 are free, setting $x_3 = s$ and $x_4 = t$, we get that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 4t \\ 3s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} = s\vec{v}_1 + t\vec{v}_2.$$

The vectors $\vec{v}_1$ and $\vec{v}_2$ span the kernel of A.

To find a basis for the image, we instead look at the leading variables. Since x_1 and x_2 are leading, the first two columns of A are linearly independent, while the others are in their span, hence the image is spanned by the vectors

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}.$$

Problem 4. Let P_2 denote the space of polynomials of degree less than or equal to two, and let T be the transformation from P_2 to P_2 be defined by formula:

$$T(f(t)) = f(2t-1)$$

e.g. if $f(t) = t^2 + t + 1$, then $T(f(t)) = (2t-1)^2 + (2t-1) + 1 = 4t^2 + 2t + 1$.

1. Show that T is a linear transformation.
2. Let $\mathcal{B} = (1, t, t^2)$ be the standard basis of P_2. Find the matrix B of T with respect to the basis $\mathcal{B}$.
3. Find a basis for the kernel of B (Hint: what is the kernel of the transformation T?)
4. Find a basis for the image of B.

Solution. To show that T is linear, we show that it preserves sums:

$$T(f(t) + g(t)) = f(2t-1) + g(2t-1) = (f+g)(2t-1) = T((f+g)(t)$$

and scalar products:

$$T(kf(t)) = kf(2t-1) = (kf)(2t-1) = T((kf)(t)).$$

The action on the standard basis is the following:

$$T(1) = 1, \quad T(t) = 2t - 1 = -1 + 2t, \quad T(t^2) = (2t-1)^2 = 1 - 4t + 4t^2,$$

so the matrix B of T with respect to $\mathcal{B}$ is

$$B = \left[\ [T(1)]_{\mathcal{B}} \quad [T(t)]_{\mathcal{B}} \quad [T(t^2)]_{\mathcal{B}}\ \right] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

To find the basis for the kernel and the image of T, we note that $\det B = 8$ is non-zero, so the matrix B is non-degenerate. Hence T has trivial kernel, and by the rank-nullity theorem the dimension of the image is 3, hence the image is all of P_2. Therefore, the image is spanned by $\mathcal{B}$.

Alternatively, the kernel of T is the set of polynomials that become zero when you substitute $2t-1$ into them, and it is clear that only the zero polynomial has this property.

Problem 5. Let V be the subspace of $\mathbb{R}^3$ defined by the equation

$$2x_1 - x_2 - x_3 = 0.$$

1. Find a basis for V. What is the dimension of V?
2. Use Gram–Schmidt orthogonalization on this basis to find an orthonormal basis for V.
3. Let $T(\vec{x}) = \text{proj}_V(\vec{x})$ be the orthogonal projection onto the space V. Find a formula for T.
4. Find the matrix A of the linear transformation T.

Solution. The space V is spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

and has dimension two. We apply Gram–Schmidt orthogonalization:

$$\vec{u}_1 = \frac{\vec{v}_1}{||\vec{v}_1||} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \quad \vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 4/5 \\ -2/5 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \frac{\vec{v}_2^{\perp}}{||\vec{v}_2^{\perp}||} = \begin{bmatrix} 2/\sqrt{30} \\ -1/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}.$$

The formula for the projection onto a space with an orthonormal basis is

$$\text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2.$$

In matrix form,

$$\text{proj}_V \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left(\frac{x_1}{\sqrt{5}} + \frac{2x_2}{\sqrt{5}} \right) \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} + \left(\frac{2x_1}{\sqrt{30}} - \frac{x_2}{\sqrt{30}} + \frac{5x_3}{\sqrt{30}} \right) \begin{bmatrix} 2/\sqrt{30} \\ -1/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix} =$$

$$= \begin{bmatrix} x_1/5 + 2x_2/5 \\ 2x_1/5 + 4x_2/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_1/30 - 2x_2/30 + 10x_3/30 \\ -2x_1/30 + x_2/30 - 5x_3/30 \\ 10x_1/30 - 5x_2/30 + 25x_3/30 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ \frac{1}{3}x_1 + \frac{5}{6}x_2 - \frac{1}{6}x_3 \\ \frac{1}{3}x_1 - \frac{1}{6}x_2 + \frac{5}{6}x_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 5/6 & -1/6 \\ 1/3 & -1/6 & 5/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Problem 6. Find the determinant of the following matrix:

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{bmatrix}$$

Solution. The easiest way to solve this problem is by row operations:

$$\begin{vmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{vmatrix} \begin{matrix} \\ -1 \cdot I \\ -1 \cdot I \\ -2 \cdot I \end{matrix} = \begin{vmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 1 \cdot 3 \cdot (-2) \cdot 4 = 24.$$

Problem 7. Let A be the matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix},$$

where a, b and c are constant numbers.

1. Find the eigenvalues of A and their algebraic multiplicities.
2. For each eigenvalue, find a basis for the corresponding eigenspace. Find the geometric multiplicites of the eigenvalues.
3. For what values of a, b and c is the matrix A diagonalizable?

Solution. The characteristic polynomial of A is

$$\det(A - \lambda I_3) = \left| \begin{matrix} 1-\lambda & a & b \\ 0 & 2-\lambda & c \\ 0 & 0 & 1-\lambda \end{matrix} \right] = (1-\lambda)(2-\lambda)(1-\lambda),$$

so the eigenvalues are $\lambda = 1$ with algebraic multiplicity 2 and $\lambda = 2$ with algebraic multiplicity 1.

The eigenspace E_2 is always one-dimensional, so we describe it first:

$$E_2 = \ker(A - 2I_3) = \begin{bmatrix} -1 & a & b \\ 0 & 0 & c \\ 0 & 0 & -1 \end{bmatrix}$$

By inspection, we see that the vector

$$\vec{v}_1 = \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}$$

is in the kernel, and hence spans E_2.

The eigenspace E_1 is the kernel of the matrix $A - I_3$, i.e. the set of solutions to the system

$$\begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The second equation of the system gives us $x_2 = -cx_3$, and plugging this into the first equation we get $(b - ac)x_3 = 0$. Here there are two possibilities. If $b - ac = 0$, then this equation is vacuous, so x_1 and x_3 are free variables, and $x_2 = -cx_3$. The geometric multiplicity is 2, and a basis for the eigenspace is then

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -c \\ 1 \end{bmatrix}.$$

If $b - ac \neq 0$, then $x_3 = 0$ and hence $x_2 = 0$. Then only x_1 is a free variable, and $\vec{v}_2$ spans the eigenspace E_1, and the geometric multiplicity is 1.

The matrix A is diagonalizable if and only if the geometric multiplicities add up to 3, i.e. if and only if $b - ac = 0$.

Problem 8. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an arbitrary 2×2 matrix.

1. Let $p(\lambda)$ denote the characteristic polynomial of A:

   $$p(\lambda) = \lambda^2 + x\lambda + y.$$

   Express x and y in terms of the coefficiens of A.

2. Evaluate the matrix.

   $$A^2 + xA + yI_2$$

3. Find a non-zero 2×2 matrix A such that A^2 is the zero matrix.

4. **Extra Credit.** Show that the does not exist a 2×2 matrix A such that A^2 is not the zero matrix, but A^3 is the zero matrix.

Solutions. The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - \lambda \mathrm{tr} A + \det A,$$

so $x = -(a+d)$ and $y = ad - bc$.

We know how to define the powers of a matrix, so we can plug a matrix into a polynomial:

$$p(A) = A^2 + xA + y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In other words, *a matrix satisfies its characteristic equation.*

For part 3, we are looking for a non-zero matrix A such that $A^2 = 0$. First of all, $\det(A^2) = (\det A)^2 = 0$, so A must be degenerate. Furthermore, A satisfies the equation $A^2 - A \cdot \mathrm{tr} A + \det A = 0$, where $A^2 = 0$ and $\det A = 0$. Therefore, $\mathrm{tr} A = 0$, and the characteristic polynomial of A is $\lambda^2 = 0$, so A

has one eigenvalue $\lambda = 0$ with algebraic multiplicity 2. An example of such a matrix is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For part 4, we note that again $\det A = 0$, so the matrix A satisfies $A^2 - A \cdot \mathrm{tr}A = 0$. Multiplying this by A we get that $A^3 = A^2 \mathrm{tr}A$. If $A^3 = 0$ and $A^2 \neq 0$, then $\mathrm{tr}A = 0$, but then we get a contradiction, since $A^2 = \mathrm{tr}A = 0$.