22)

$$A = \begin{pmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & -15 \\ 0 & -5 & -25 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$ 3

Since $\begin{pmatrix} -6 \\ 5 \\ 0 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\therefore$ $\begin{pmatrix} -6 \\ 5 \\ 0 \end{pmatrix}$ is in span$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

$\therefore$ a basis of Image$(A) = \left\{\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix}\right\}$ 3

To find a basis for the ker,

$$\begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} a_1 - 6a_3 = 0 \\ a_2 + 5a_3 = 0 \\ - \end{cases}$$ let $a_3 = t$

$\Rightarrow \begin{pmatrix} 6t \\ -5t \\ t \end{pmatrix}$

$\therefore$ Ker $(A) = \left\{\begin{pmatrix} 6t \\ -5t \\ t \end{pmatrix} : t \in \mathbb{R}\right\}$ 4

We can take ~~the~~ any t to be the basis,

by convention let $t = 1$, the basis is

$\left\{\begin{pmatrix} 6 \\ -5 \\ 1 \end{pmatrix}\right\}$

//

28) let $a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$ $a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$

$a_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 4 \end{pmatrix}$ $a_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ k \end{pmatrix}$

Notice that $2a_1 + 3a_2 + 4a_3$

$$= \begin{pmatrix} 2 \\ 3 \\ 4 \\ 9 \end{pmatrix}$$

and we have $a_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ k \end{pmatrix}$

Also, $2a_1 + 3a_2 + 4a_3$ is the only possible combination to make the first 3 entries to be $\begin{pmatrix} 2 \\ 3 \\ 4 \\ * \end{pmatrix}$

$\therefore$ $a_4 \in \text{span}\{a_1, a_2, a_3\}$ if and only if

$$k = 29$$

$\therefore$ $\forall\, k \in \mathbb{R} \setminus \{29\}$, these vectors form a basis for $\mathbb{R}^4$

//

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$\therefore$ $\text{Ker}(A) = \left\{\begin{pmatrix} 6t \\ -5t \\ t \end{pmatrix} : t \in \mathbb{R}\right\}$ 4

We can take ~~the~~ any t to be the basis,

by convention let $t = 1$, the basis is

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//

30) Since we only have one equation,

let $x_1 = s$, $x_3 = r$, $x_4 = t$.

2

then $x_2 = 2s + 2r + 4t$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s \\ 2s+2r+4t \\ r \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$ 6

Since $s, r, t \in \mathbb{R}$ are arbitrary 2

the basis is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\}$

(*) Other answers are acceptable.