MAT 211, Spring 2012
Solutions to Homework Assignment 8

Maximal grade for HW8: 100 points

Section 5.1. 2. (10 points) Find the length of a vector $v = (2, 3, 4)$.

Answer: $\sqrt{29}$.

Solution: The length of a vector can be found using a formula

$$v \cdot v = ||v||^2 \quad \Longleftrightarrow \quad ||v|| = \sqrt{v \cdot v}.$$

We have

$$||v|| = \sqrt{v \cdot v} = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}.$$

4. (20 points) Find the angle between the vectors $u = (1, 1)$ and $v = (7, 11)$.

Answer: $\alpha = \arccos\left(\frac{9}{\sqrt{85}}\right)$.

Solution: The angle between two vectors can be found using a formula

$$u \cdot v = ||u|| \cdot ||v|| \cdot \cos(\alpha) \quad \Longleftrightarrow \quad \alpha = \arccos\left(\frac{u \cdot v}{||u|| \cdot ||v||}\right).$$

We have

$$u \cdot v = 1 \cdot 7 + 1 \cdot 11 = 18, \quad ||u|| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad ||v|| = \sqrt{7^2 + 11^2} = \sqrt{170},$$

hence

$$\alpha = \arccos\left(\frac{18}{\sqrt{2}\sqrt{170}}\right) = \arccos\left(\frac{9}{\sqrt{85}}\right).$$

6. (20 points) Find the angle between the vectors $u = (1, -1, 2, -2)$ and $v = (2, 3, 4, 5)$.

Answer: $\alpha = \arccos\left(\frac{-1}{2\sqrt{15}}\right)$.

Solution: We have

$$u \cdot v = 1 \cdot 2 - 1 \cdot 3 + 2 \cdot 4 - 2 \cdot 5 = -3, \quad ||u|| = \sqrt{1^2 + (-1)^2 + 2^2 + (-2)^2} = \sqrt{10},$$

$$||v|| = \sqrt{2^2 + 3^2 + 4^2 + 5^2} = \sqrt{54}.$$

Therefore

$$\alpha = \arccos\left(\frac{u \cdot v}{||u|| \cdot ||v||}\right) = \arccos\left(\frac{-3}{\sqrt{10}\sqrt{54}}\right) = \arccos\left(\frac{-1}{2\sqrt{15}}\right).$$

10. (10 points) For which values of the constant k are the vectors $u = (2, 3, 4)$ and $v = (1, k, 1)$ perpendicular?

Answer: $k = -2$.

Solution: The vectors u and v are perpendicular, if

$$u \cdot v = 0 \quad \iff \quad 2 + 3k + 4 = 0 \quad \iff \quad k = -2.$$

11. (20 points) Consider the vectors $u = (1, 1, \ldots, 1)$ and $v = (1, 0, \ldots, 0)$ in $\mathbb{R}^n$.

a) Find the angle θ between them for $n = 2, 3, 4$.

b) Find the limit of θ as n approaches infinity.

Answer: $\theta = \arccos(\frac{1}{\sqrt{n}}), \lim_{n\to\infty} \theta(n) = \frac{\pi}{2}$.

Solution: We have

$$u \cdot v = 1, \quad ||u|| = \sqrt{1^2 + \ldots + 1^2} = \sqrt{n}, \quad ||v|| = \sqrt{1} = 1,$$

so $\theta = \arccos(\frac{1}{\sqrt{n}})$. As n approaches infinity, $\frac{1}{\sqrt{n}}$ approaches 0, so $\theta(n)$ approaches $\arccos(0) = \pi/2$.

For $n = 2$ we have $\theta = \arccos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$, for $n = 3$ $\theta = \arccos(\frac{1}{\sqrt{3}})$, for $n = 4$ $\theta = \arccos(\frac{1}{2}) = \frac{\pi}{3}$.

15. (20 points) Consider the vectors $v = (1, 2, 3, 4)$ in $\mathbb{R}^4$. Find a basis of the subspace of $\mathbb{R}^4$ consisting of all vectors perpendicular to v.

Solution: Let $u = (u_1, u_2, u_3, u_4)$ be a vectors from this subspace, then

$$u \perp v \quad \Longleftrightarrow \quad u \cdot v = 0 \quad \Longleftrightarrow \quad u_1 + 2u_2 + 3u_3 + 4u_4 = 0.$$

This a linear equation on u_i, and we can choose u_2, u_3, u_4 as free parameters and compute $u_1 = -2u_2 - 3u_3 - 4u_4$. Therefore the basis in the space of solutions is given by the vectors $(-2, 1, 0, 0), (-3, 0, 1, 0), (-4, 0, 0, 1)$.