MAT 211, Spring 2012
Solutions to Homework Assignment 7

Maximal grade for HW7: 100 points

Section 3.2. Find a basis of the image of the matrices.
27. (10 points)

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Solution: Let us transform the matrix to the reduced row-echelon form. Let us subtract the first row from the second and third:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

Let us subtract the second row multiplied by 2 from the third:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We conclude that the rank of the matrix equals to 2, and none of its columns is redundant. Therefore the basis in the image: $(1,1,1),(1,2,3)$.

28.(10 points)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{pmatrix}.$$

Solution: Let us transform the matrix to triangular form. Let us subtract the first row from the second and third:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 2 & 6 \end{pmatrix}.$$

Let us subtract the second row multiplied by 2 from the third:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{pmatrix}.$$

We conclude that the rank of the matrix equals to 3, and none of its columns is redundant. Therefore the basis in the image: $(1,1,1), (1,2,3), (1,5,7)$.

31. (10 points)

$$\begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{pmatrix}.$$

Solution: Let us transform the matrix to the reduced row-echelon form. Let us subtract the first row from all other rows:

$$\begin{pmatrix} 1 & 5 \\ 0 & -4 \\ 0 & -8 \\ 0 & -17 \end{pmatrix}.$$

Divide the second row by -4, the third by -8 and the third by -17:

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Subtract the second row from third and fourth:

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We conclude that the rank of the matrix equals to 2, and none of its columns is redundant. Therefore the basis in the image: $(1, 2, 3, 5), (5, 6, 7, 8)$.

34. (10 points) Consider the 5×4 matrix A with columns v_1, v_2, v_3, v_4. We are told that the vector $(1, 2, 3, 4)$ is in the kernel of A. Write v_4 as a linear combination of v_1, v_2, v_3, v_4.

Solution: By definition of the kernel, we have $v_1 + 2v_2 + 3v_3 + 4v_4 = 0$, therefore

$$v_4 = -\frac{1}{4}v_1 - \frac{1}{2}v_2 - \frac{3}{4}v_3.$$

52. (10 points) For which values of the constants a, b, c, d, e, f are the vectors $(a, 0, 0, 0), (b, c, 0, 0), (d, e, f, 0)$ linearly independent?

Answer: $a \neq 0, c \neq 0$ and $f \neq 0$.

Solution: They are linearly independent if the following matrix has rank 3:

$$\begin{pmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix}.$$

We have the following cases:

1) $a \neq 0, c \neq 0$ and $f \neq 0$. Let us divide the first row by a, the second by c and the third by f:

$$\begin{pmatrix} 1 & b/a & d/a \\ 0 & 1 & e/c \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that the rank equals to 3.

2) $a = 0$. In this case the first column is redundant.

3) $a \neq 0, c = 0$ In this case we have a leading 1 in the first column in the reduced row-echelon form, but we cannot have a leading 1 in the second column. Therefore the second column is redundant.

4) $a \neq 0, c \neq 0, f = 0$. In this case we have a leading 1 in the first and second columns after division by a and c, but we cannot have a leading 1 in the third column. Therefore the third column is redundant.

Section 3.3. 27. (10 points) Determine whether the following vectors form a basis of $\mathbb{R}^4$:

$$(1, 1, 1, 1), (1, -1, 1, -1), (1, 2, 4, 8), (1, -2, 4, -8).$$

Solution: We have to compute the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 8 & -8 \end{pmatrix}.$$

Subtract the first row from all other rows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{pmatrix}.$$

Subtract the second row from the fourth:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{pmatrix}.$$

Subtract the third row, multiplied by 2, from the fourth row:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{pmatrix}.$$

We see that the matrix has rank 4, so there are no redundant vectors and the original vectors were independent, therefore they form a basis of $\mathbb{R}^4$.

29. (10 points) Find a basis of the subspace of $\mathbb{R}^3$ defined by the equation $2x_1 + 3x_2 + x_3 = 0$.

Solution: The values of x_2 and x_3 can be chosen arbitrarily, and $x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3$. We can set $x_2 = 1$ and $x_3 = 0$, then $x_1 = -\frac{3}{2}$, or $x_2 = 0$ and $x_3 = 1$, then $x_1 = -\frac{1}{2}$ Therefore the basis in the subspace is $(-\frac{3}{2}, 1, 0), (-\frac{1}{2}, 0, 1)$.

36. (10 points) Can you find a 3×3 matrix A such that $Im(A) = Ker(A)$? Explain.

Solution: By the Rank-Nullity theorem we have $\dim Im(A) + \dim Ker(A) = 3$. Therefore if $Im(A) = Ker(A)$ then $\dim Im(A) = 3/2$, what is impossible. Therefore such a matrix does not exist.

37. (10 points) Give an example of a 4×5 matrix A with $\dim Ker(A) = 3$.

Solution: Such a matrix should have 5 columns, so its rank equals to $5 - 3 = 2$. An example of a rank 2 matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

38. (10 points) a) Consider a linear transformation T from $\mathbb{R}^5$ to $\mathbb{R}^3$. What are the possible values of $\dim Ker(T)$?

b) Consider a linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^7$. What are the possible values of $\dim Im(T)$?

Solution: a) The rank of a 3×5 matrix is bounded by 3, so it can be equal to 0,1,2,3. Therefore $\dim Ker(T) = 5 - rk(T)$ can be equal to 5,4,3,2.

b) The rank of a 7×4 matrix is bounded by 4, so it can be equal to 0,1,2,3,4. Therefore $\dim Im(T) = rk(T)$ can be equal to 0,1,2,3,4.