MAT 211, Spring 2012
Solutions to Homework Assignment 3

Maximal grade for HW3: 100 points

Section 1.3. 4. (10 points) Find the rank of the matrix:

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Answer: 2.

Solution: Let us transform this matrix to the triangular form. Subtract from 2nd row 1st, multiplied by 2, and from 3rd 1st, multiplied by 3:

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

Divide 2nd row by (-3) and 3rd by (-6):

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

Subtract 2nd row from 3rd:

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

There are two leading 1's, so the rank equals to 2.

Section 2.1. 4. (10 points) Find the matrix of linear transformation:

$$y_1 = 9x_1 + 3x_2 - 3x_3$$

$$y_2 = 2x_1 - 9x_2 + x_3$$

$$y_3 = 4x_1 - 9x_2 - 2x_3$$

$$y_4 = 5x_1 + x_2 + 5x_3.$$

Solution: Let us compute the values of the transformation on basic vectors:

$$T(1,0,0) = (9,2,4,5), \quad T(0,1,0) = (3,-9,-9,1), \quad T(0,0,1) = (-3,1,-2,5).$$

We have to write these values in the rows of our matrix:

$$A = \begin{pmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{pmatrix}.$$

6. (10 points) Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

Is this transformation linear? If so, find its matrix.

Solution: It is easy to check that $T(v_1+v_2) = T(v_1)+T(v_2)$, $\quad T(\alpha\cdot v) = \alpha \cdot T(v)$, so T is a linear transformation. To find its matrix let us compute its values on the basic vectors:

$$T(1,0) = (1,2,3), \quad T(0,1) = (4,5,6).$$

We have to write these values in the rows of our matrix:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

44. (10 points) The cross product of two vectors in $\mathbb{R}^3$ is given by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Consider and arbitrary vector v in $\mathbb{R}^3$. Find the matrix of the linear transformation

$$T(x) = v \times x.$$

Solution: Let us compute the values of the transformation on basic vectors:

$$T(1,0,0) = \begin{pmatrix} v_2 \cdot 0 - v_3 \cdot 0 \\ v_3 \cdot 1 - v_1 \cdot 0 \\ v_1 \cdot 0 - v_2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ v_3 \\ -v_2 \end{pmatrix},$$

$$T(0,1,0) = \begin{pmatrix} v_2 \cdot 0 - v_3 \cdot 1 \\ v_3 \cdot 0 - v_1 \cdot 0 \\ v_1 \cdot 1 - v_2 \cdot 0 \end{pmatrix} = \begin{pmatrix} -v_3 \\ 0 \\ v_1 \end{pmatrix},$$

$$T(0,0,1) = \begin{pmatrix} v_2 \cdot 1 - v_3 \cdot 0 \\ v_3 \cdot 0 - v_1 \cdot 1 \\ v_1 \cdot 0 - v_2 \cdot 0 \end{pmatrix} = \begin{pmatrix} v_2 \\ -v_1 \\ 0 \end{pmatrix},$$

We have to write these values in the rows of our matrix:

$$A = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

45. (10 points) Consider two linear transformations $y = T(x)$ and $z = L(y)$. Is the transformation $z = L(T(x))$ linear as well?

Solution: We have

$$L(T(x_1+x_2)) = L(T(x_1)+T(x_2)) = L(T(x_1))+L(T(x_2)), \quad L(T(\alpha \cdot x)) = L(\alpha \cdot T(x)) = \alpha L(T(x)),$$

so the composite transform is linear.

46. (10 points) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Find the matrix of the linear transform $T(x) = B(A(x))$.

Solution: Let us compute the values of the transformation on basic vectors:

$$T(1,0) = B(A(1,0)) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} pa+qc \\ ra+sc \end{pmatrix},$$

$$T(0,1) = B(A(0,1)) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} pb+qd \\ rb+sd \end{pmatrix}.$$

We have to write these values in the rows of our matrix:

$$A = \begin{pmatrix} pa+qc & pb+qd \\ ra+sc & rb+sd \end{pmatrix}.$$

Section 2.3. Compute matrix products.

4. (10 points)

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}.$$

Answer:

$$\begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{pmatrix}.$$

7. (10 points)

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

Answer:

$$\begin{pmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{pmatrix}.$$

14. (10 points) For the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad E = (5)$$

Compute all possible matrix products.

Solution: A product of two matrices is defined, if the number of columns in the first one equals to the number of rows in the second one. Therefore the only defined products are $AA, BC, BD, CC, CD, DE, EB$. Let us compute them:

$$AA = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad BC = \begin{pmatrix} 14 & 8 & 2 \end{pmatrix},$$

$$BD = \begin{pmatrix} 6 \end{pmatrix}, \quad CC = \begin{pmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{pmatrix},$$

$$CD = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}, \quad DE = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix},$$

$$EB = \begin{pmatrix} 5 & 10 & 15 \end{pmatrix}, \quad DB = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

$$EE = (25).$$

29b. (10 points) Let

$$D_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Find $D_\alpha D_\beta$ and $D_\beta D_\alpha$.

Solutions: We have

$$D_\alpha D_\beta = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} =$$

$$\begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

Analogously $D_\beta D_\alpha = D_\alpha D_\beta$.