# MAT 211, Spring 2012
# Solutions to Homework Assignment 2

**Maximal grade for HW2: 100 points**

**Section 1.2**. Use Gauss-Jordan elimination to solve linear systems.
4.(10 points)

$$\begin{cases} x + y = 1 \\ 2x - y = 5 \\ 3x + 4y = 2 \end{cases}$$

**Answer:** $x = 2, y = -1$.

**Solution:** Let us apply the Gauss-Jordan elimination to the matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 3 & 4 & 2 \end{array}\right)$$

Subtract from the 2nd row 1st multiplied by 2 and from the 3rd the 1st multiplied by 3:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{array}\right)$$

Divide the 2nd row by $(-3)$ and subtract from the 3rd:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

Subtract the 2nd equation from the 1st:

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore $x = 2, y = -1$.

8. (10 points)

$$\begin{cases} x_2 + 2x_4 + 3x_5 & = 0 \\ 4x_4 + 8x_5 & = 0 \end{cases}$$

**Answer:** $x_1, x_3, x_5$ are arbitrary, $x_2 = x_5, x_4 = -2x_5$.

**Solution:** Let us apply the Gauss-Jordan elimination to the matrix:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 & 8 & 0 \end{array}\right)$$

Divide the 2nd row by 4:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

Subtract from 1st row the 2nd multiplied by 2:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

Therefore $x_1, x_3, x_5$ are arbitrary, $x_2 = x_5, x_4 = -2x_5$.

11. (20 points)

$$\begin{cases} x_1 + 2x_3 + 4x_4 & = -8 \\ x_2 - 3x_3 - x_4 & = 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 & = 0 \\ -x_2 + 3x_3 + 4x_4 & = -12 \end{cases}$$

**Answer:** $x_3$ is arbitrary, and $x_1 = -2x_3, x_2 = 3x_3 + 4, x_4 = -2$.

**Solution:** Let us apply the Gauss-Jordan elimination to the matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right)$$

Subtract from the 3nd row the 1st multiplied by 3:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right)$$

Divide 3rd row by 4 and multiply the 4th row by (-1):

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 1 & -3 & -4 & 12 \end{array}\right)$$

Subtract the 2nd row from 3rf and 4th:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 6 \end{array}\right)$$

Divide 4th row by (-3) and swap the 3rd and 4th row:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Subtract the 3th row from 1th and 2nd:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore $x_3$ is arbitrary, and $x_1 = -2x_3, x_2 = 3x_3 + 4, x_4 = -2$.

16. (20 points)

$$\begin{cases} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 & = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 & = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 & = 11 \end{cases}$$

**Answer:** $x_2, x_3, x_5$ are arbitrary, $x_1 = 6 - 2x_2 - 3x_3 - 5x_5, x_4 = 7 - 2x_5$.

**Solution:** Let us apply the Gauss-Jordan elimination to the matrix:

$$\left(\begin{array}{ccccc|c} 3 & 6 & 9 & 5 & 25 & 53 \\ 7 & 14 & 21 & 9 & 53 & 105 \\ -4 & -8 & -12 & 5 & -10 & 11 \end{array}\right)$$

Add the 3rd row to the 1st and change the sign:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -10 & -15 & -64 \\ 7 & 14 & 21 & 9 & 53 & 105 \\ -4 & -8 & -12 & 5 & -10 & 11 \end{array}\right)$$

Subtract the 1st row multiplied by 7 from 2nd and add 1st multiplied by 4 to 3rd:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -10 & -15 & -64 \\ 0 & 0 & 0 & 79 & 158 & 553 \\ 0 & 0 & 0 & -35 & -70 & -245 \end{array}\right)$$

Divide 2nd row by 79 and 3rd by 35:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -10 & -15 & -64 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & -1 & -2 & -7 \end{array}\right)$$

Add 2nd row to the 3rd:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -10 & -15 & -64 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Add 2nd, multiplied by 10, to 1st:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore $x_2, x_3, x_5$ are arbitrary, $x_1 = 6 - 2x_2 - 3x_3 - 5x_5, x_4 = 7 - 2x_5$.

24. (10 points) Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A?

**Solution:** Let us consider three types of elementary transformations and present inverse operations for them:

1) Swap two rows ; Inverse operation - same swap.

2) Multiply a row by $\alpha$; Inverse operation - divide by $\alpha$.

3) Add $i$th row multiplied by $\alpha$ to $k$th row; Inverse operation - subtract $i$th row multiplied by $\alpha$ from $k$th row.

29. (10 points) Consider the chemical reaction

$$aNO_2 + bH_2O \to cHNO_2 + dHNO_3.$$

Balance this reaction.

**Answer:** $2NO_2 + H_2O \to HNO_2 + HNO_3$.

**Solution:** We have a linear system

$$\begin{cases} a = c + d & (N) \\ 2b = c + d & (H) \\ 2a + b = 2c + 3d & (O) \end{cases}$$

Let us substitute $a$ and $b$ to the third equation:

$$2c + 2d + \frac{1}{2}c + \frac{1}{2}d = 2c + 3d \Leftrightarrow \frac{1}{2}c = \frac{1}{2}d.$$

Therefore $c = d$, so $a = 2d, b = d$.

45. (20 points) How many solutions does this system have?

$$\begin{cases} x + 2y + 3z & = 4 \\ x + ky + 4z & = 6 \\ x + 2y + (k+2)z & = 6 \end{cases}$$

**Answer:** No solutions for $k = 1$, infinitely many for $k = 2$, one for all other $k$.

**Solution:**Let us apply the Gauss-Jordan elimination to the matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & k & 4 & 6 \\ 1 & 2 & (k+2) & 6 \end{array}\right)$$

Subtract 1st row from 2nd and 3rd:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & k-2 & 1 & 2 \\ 0 & 0 & (k-1) & 2 \end{array}\right)$$

If $k = 1$, the last equation has a form $0 = 2$, so there are no solutions. If $k = 2$, the matrix has a form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right),$$

if we subtract 2nd row from the 3rd, we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so there are infinitely many solutions. For all other $k$ the system has rank 3, so there is a unique solution.