MAT 211, Spring 2012
Solutions to Homework Assignment 12

Maximal grade for HW12: 100 points

Section 6.1 Find the determinants and find out if a matrix is invertible

6. (10 points)

$$\begin{pmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

Solution: This is a lower-triangular matrix, so its determinant equals to the product of its diagonal entries: $\det A = 6 \cdot 4 \cdot 1 = 24$. Since $\det A \neq 0$, the matrix is invertible.

8. (20 points)

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

Solution: 1) Let us subtract the first row from the second, and the first, multiplied by 3, from the third row:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -4 & -8 \end{pmatrix}.$$

Subtract the second row, multiplied by 4, from the third:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the matrix has rank 2, so it is not invertible and its determinant vanishes.

2)

$$\det A = 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} =$$

$$1(1-2) - 2(1-3) + 3(2-3) = -1 + 4 - 3 = 0.$$

12. (10 points) Determine all k such that the matrix is invertible: $\begin{pmatrix} 1 & k \\ k & 4 \end{pmatrix}$.

Solution: The determinant of this matrix equals to $4-k^2$, so it vanishes at $k = \pm 2$. Therefore the matrix is invertible, if $k \neq 2$ and $k \neq (-2)$.

21. (20 points) Determine all k such that the matrix is invertible: $\begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$.

Solution:

$$\det A = k \cdot \begin{vmatrix} k & 1 \\ 1 & k \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & k \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & k \\ 1 & 1 \end{vmatrix} =$$

$$k(k^2-1)-(k-1)+(1-k) = k(k-1)(k+1)-2(k-1) = (k^2+k-2)(k-1) = (k-1)(k+2)(k-1).$$

Therefore the matrix is invertible, if $k \neq 1$ and $k \neq (-2)$.

Section 6.2 29. (20 points) Let P_n denote the $n \times n$ matrix whose entries are all ones, except the zeroes directly below the main diagonal. Find the determinant of P_n.

Answer: $\det P_n = 1$.

Solution: Let us apply the Gauss-Jordan elimination to P_n. The process is similar for all n, let us illustrate it for $n = 5$:

$$P_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Subtract the second row from the first:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Subtract the first row from all other rows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Remark that we get 1 in the top left corner and P_4 in the complement to it. Let us repeat the same procedures with P_4:

Subtract the third row from the second:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Subtract the second row from all further rows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Subtract the 4th row from the 3rd:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Subtract the 3rd row from all further rows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Subtract the 5th row from the 4th:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We get a unit matrix with the determinant 1. Since we used the elementary transformations of the 3rd type only, they did not change the determinant and $\det P_5 = \det P_n = 1$.

Section 6.3 22. (20 points) Solve the linear system using Cramer's rule:

$$\begin{cases} 3x + 7y & = 1 \\ 4x + 11y = 3 \end{cases}$$

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Solution: We have

$$x = \frac{\begin{vmatrix} 1 & 7 \\ 3 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 7 \\ 4 & 11 \end{vmatrix}} = \frac{11 - 21}{33 - 28} = -\frac{10}{5} = -2,$$

$$y = \frac{\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 7 \\ 4 & 11 \end{vmatrix}} = \frac{9 - 4}{33 - 28} = \frac{5}{5} = 1.$$