MAT 211, Spring 2012
Solutions to Homework Assignment 11

Maximal grade for HW11: 100 points

Section 5.3. Suppose that the matrices A and B are orthogonal. Which of these matrices are orthogonal as well?

5. (10 points) $3A$

Solution: Since A is orthogonal, $A^T A = I$. Therefore

$$(3A)^T \cdot (3A) = 9A^T A = 9I,$$

so $3A$ is not orthogonal.

6. (10 points) $-B$

Solution: Since B is orthogonal, $B^T B = I$. Therefore

$$(-B)^T \cdot (-B) = B^T B = I,$$

so $-B$ is orthogonal.

7. (10 points) AB

Solution: Since A and B are orthogonal, $A^T A = B^T B = I$. Therefore

$$(AB)^T \cdot (AB) = B^T A^T AB = B^T B = I,$$

so AB is orthogonal.

40. (15 points) Consider the subspace W of $\mathbb{R}^4$ spanned by the vectors $v_1 = (1, 1, 1, 1)$ and $v_2 = (1, 9, -5, 3)$. Find the matrix of the orthogonal projection onto W.

Solution: Let us find the orthonormal basis in W by Gram-Schmidt process. The length of v_1 equals to $||v_1|| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$. Therefore

$$u_1 = v_1/||v_1|| = (1/2, 1/2, 1/2, 1/2).$$

Now

$$v_2^{||} = (v_2 \cdot u_1)u_1 = (1/2 + 9/2 - 5/2 + 3/2)u_1 = 4u_1 = (2, 2, 2, 2),$$

$$v_2^{\perp} = v_2 - v_2^{\perp} = (1, 9, -5, 3) - (2, 2, 2, 2) = (-1, 7, -7, 1),$$

$$||v_2^{\perp}|| = \sqrt{(-1)^2 + 7^2 + (-7)^2 + 1^2} = \sqrt{100} = 10,$$

so

$$u_2 = v_2^{\perp}/||v_2^{\perp}|| = (-1/10, 7/10, -7/10, 1/10).$$

Therefore the matrix Q with columns u_1 and u_2 has a form

$$Q = \begin{pmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{pmatrix},$$

and the matrix of the projection onto W equals to

$$QQ^T = \begin{pmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/10 & 7/10 & -7/10 & 1/10 \end{pmatrix} = \begin{pmatrix} 13/50 & 9/50 & 8/25 & 6/25 \\ 9/50 & 37/50 & -6/25 & 8/25 \\ 8/25 & -6/25 & 37/50 & 9/50 \\ 6/25 & 8/25 & 9/50 & 13/50 \end{pmatrix}$$

Section 5.4. 19. (10 points) Find the least-squares solution of the system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $b = (1, 1, 1)$.

Solution: We have to solve the equation $A^T Ax = A^T b$:

$$A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore $x^* = (1, 1)$.

29. (15 points) Find the least-squares solution of the system $Ax = B$, where

$$A = \begin{pmatrix} 1 & 1 \\ 10^{-10} & 0 \\ 0 & 10^{-10} \end{pmatrix}$$

and $b = (1, 10^{-10}, 10^{-10})$.

Solution: We have to solve the equation $A^T Ax = A^T b$:

$$A^T A = \begin{pmatrix} 1 & 10^{-10} & 0 \\ 1 & 0 & 10^{-10} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 10^{-10} & 0 \\ 0 & 10^{-10} \end{pmatrix} = \begin{pmatrix} 1 + 10^{-20} & 1 \\ 1 & 1 + 10^{-20} \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 10^{-10} & 0 \\ 1 & 0 & 10^{-10} \end{pmatrix} \begin{pmatrix} 1 \\ 10^{-10} \\ 10^{-10} \end{pmatrix} = \begin{pmatrix} 1 + 10^{-20} \\ 1 + 10^{-20} \end{pmatrix}$$

It is easy to check that the solution to this system is

$$x = \left(\frac{1 + 10^{-20}}{2 + 10^{-20}}, \frac{1 + 10^{-20}}{2 + 10^{-20}} \right) \approx (\frac{1}{2}, \frac{1}{2}).$$

30. (15 points) Fit a linear function of the form $f = c_0 + c_1 t$ to the data points $(0, 0), (0, 1), (1, 1)$ using least squares.

Solution: We have to solve a linear system

$$\begin{cases} c_0 + 0c_1 & = 0 \\ c_0 + 0c_1 & = 1 \\ c_0 + 1c_1 & = 1 \end{cases}$$

using least squares method. Its matrix equal to

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and $b = (0,1,1)$. We have to solve the equation $A^T Ax = A^T b$:

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore

$$\begin{cases} 3c_0 + c_1 & = 2, \\ c_0 + c_1 & = 1, \end{cases}$$

so $c_0 = 1/2, c_1 = 1/2$.

31. (15 points) Fit a linear function of the form $f = c_0 + c_1 t$ to the data points $(0,3), (1,3), (1,6)$ using least squares.

Solution: We have to solve a linear system

$$\begin{cases} c_0 + 0c_1 & = 3 \\ c_0 + 1c_1 & = 3 \\ c_0 + 1c_1 & = 6 \end{cases}$$

using least squares method. Its matrix equal to

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $b = (3,3,6)$. We have to solve the equation $A^T Ax = A^T b$:

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \end{pmatrix}$$

Therefore

$$\begin{cases} 3c_0 + 2c_1 & = 12, \\ 2c_0 + 2c_1 & = 9, \end{cases}$$

so $c_0 = 3, c_1 = 3/2$.