MAT 211, Spring 2012
Solutions to Homework Assignment 10

Maximal grade for HW10: 100 points

Section 5.2. Perform the Gram-Schmidt process for the following systems of vectors.

1. (10 points) $v = (2, 1, -2)$.

Solution: We have

$$||v|| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{9} = 3,$$

so the corresponding unit vector will be

$$u = v/||v|| = (2/3, 1/3, -2/3).$$

2. (10 points) $v_1 = (6, 3, 2), v_2 = (2, -6, 3)$.

Solution: We have

$$v_1 \cdot v_1 = 6^2 + 3^2 + 2^2 = 49, \quad v_2 \cdot v_2 = 49, v_1 \cdot v_2 = 12 - 18 + 6 = 0.$$

Therefore the corresponding orthonormal basis will be

$$u_1 = v_1/7 = (6/7, 3/7, 2/7), \quad u_2 = v_2/7 = (2/7, -6/7, 3/7).$$

6. (15 points) $v_1 = (2, 0, 0), v_2 = (3, 4, 0), v_3 = (5, 6, 7)$.

Solution: We have

$$u_1 = v_1/||v_1|| = (1, 0, 0),$$

Now we can compute the second vector:

$$v_2^{||} = (v_2 \cdot u_1)u_1 = 3u_1 = (3, 0, 0), \quad v_2^{\perp} = v_2 - v_2^{||} = (0, 4, 0), \quad u_2 = v_2^{\perp}/||v_2^{\perp}|| = (0, 1, 0).$$

And the third vector:

$$v_3^{||} = (v_3 \cdot u_1)u_1 + (v_3 \cdot u_2)u_2 = 5u_1 + 6u_2 = (5, 6, 0), \quad v_3^{\perp} = v_3 - v_3^{||} = (0, 0, 7),$$

$$u_3 = v_3^{\perp}/||v_3^{\perp}|| = (0, 0, 1).$$

To sum up,

$$u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1).$$

10. (15 points) $v_1 = (1, 1, 1, 1), v_2 = (6, 4, 6, 4)$.

Solution: We have

$$||v_1||^2 = 1 + 1 + 1 + 1 = 4, \quad u_1 = v_1/||v_1|| = (1/2, 1/2, 1/2, 1/2).$$

Now

$$v_2^{||} = (v_2 \cdot u_1)u_1 = (3 + 2 + 3 + 2)u_1 = 10u_1 = (5, 5, 5, 5),$$

$$v_2^{\perp} = v_2 - v_2^{||} = (6, 4, 6, 4) - (5, 5, 5, 5) = (1, -1, 1, -1), \quad u_2 = v_2^{\perp}/||v_2^{\perp}|| = (1/2, -1/2, 1/2, -1/2).$$

To sum up,

$$u_1 = (1/2, 1/2, 1/2, 1/2), \quad u_2 = (1/2, -1/2, 1/2, -1/2).$$

Section 5.3. Verify if a matrix is orthogonal.

1. (10 points)

$$A = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}.$$

Solution: We have to check that $A^t \cdot A = I$. We have

$$A^T \cdot A = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} = \begin{pmatrix} 1 & 0.96 \\ 0.96 & 1 \end{pmatrix},$$

so the matrix A is not orthogonal.

2. (10 points)

$$A = \begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}.$$

Solution: We have to check that $A^t \cdot A = I$. We have

$$A^T \cdot A = \begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix} \cdot \begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so the matrix A is orthogonal.

37. (15 points) Is there an orthogonal transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}?$$

Solution: Let $v_1 = (2,3,0), v_2 = (-3,2,0)$. Then $v_1 \cdot v_2 = 0$, but

$$T(v_1) \cdot T(v_2) = (3,0,2) \cdot (2,-3,0) = 6.$$

Since orthogonal transformations should preserve the scalar product of vectors, T cannot be orthogonal.

41. (15 points) Find the matrix A of the orthogonal projection onto the line in $\mathbb{R}^n$ spanned by the vector $v = (1,1,\ldots,1)$.

Solution: We have

$$||v||^2 = 1 + 1 + \ldots + 1 = n, \quad ||v|| = \sqrt{n},$$

therefore to get an unit vector we have to consider

$$u = v/||v|| = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}).$$

A projection of a vector $x = (x_1, \ldots x_n)$ can be computed as follows:

$$x \cdot u = \frac{x_1}{\sqrt{n}} + \ldots + \frac{x_n}{\sqrt{n}},$$

$$x^{||} = (x \cdot u)u = (\frac{x_1}{n} + \ldots + \frac{x_n}{n}, \ldots, \frac{x_1}{n} + \ldots + \frac{x_n}{n}),$$

so the matrix of the projection has a form

$$P = \begin{pmatrix} \frac{1}{n} & \ldots & \frac{1}{n} \\ \vdots & \vdots & \vdots \\ \frac{1}{n} & \ldots & \frac{1}{n} \end{pmatrix}$$