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**Introduction to Linear Algebra**
MAT 211
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# Solutions for Midterm I

**Problem 1.** Solve the system using the Gauss-Jordan elimination and verify your answer

$$\begin{cases} x_1 \quad\quad + 2x_3 - x_4 = 1 \\ \quad x_2 \quad\quad + 2x_4 = -1 \\ x_1 - x_2 + 2x_3 - 3x_4 = 2. \end{cases}$$

**Solution.** Elementary row transformations of the augmented matrix of the system give rise to the reduced row-echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 2 & -1 \\ 1 & -1 & 2 & -3 & 2 \end{array}\right) \underset{R3-R1}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & -1 & 0 & -2 & 1 \end{array}\right) \underset{R3+R2}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Obviously, the rank of the matrix is 2. The number of free variables is $4 - 2 = 2$ (the number of unknowns minus the rank). We write down the solution starting from the back:
$x_4 = t$ (choose $x_4$ as a free variable),
$x_3 = s$ (choose $x_3$ as a free variable),
$x_2 = -1 - 2t$ (from the second row in the rref),
$x_1 = 1 + t - 2s$ (from the first row in the rref).

So the solution is

$$(x_1, x_2, x_3, x_4) = (1 + t - 2s, -1 - 2t, s, t) = (1, -1, 0, 0) + t(1, -2, 0, 1) + s(-2, 0, 1, 0),$$

where $t$ and $s$ are arbitrary real numbers. Geometrically, the solution is a plane in $\mathbb{R}^4$.

To verify the solution, we substitute it into the three equations of the system:

$$\begin{cases} 1 + t - 2s + 2s - t & = 1 \\ -1 - 2t + 2t & = -1 \\ 1 + t - 2s + 1 + 2t + 2s - 3t & = 2 \end{cases}.$$

Since all the equations are satisfied, our solution is correct.

**Answer:** $\boxed{(x_1, x_2, x_3, x_4) = (1 + t - 2s, -1 - 2t, s, t), \quad t,\ s \in \mathbb{R}.}$

**Problem 2.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal projection onto the line $x - 2y = 0$ followed by a counterclockwise rotation by $45°$. Find a matrix of $T$ (with respect to the standard basis). Describe geometrically and show on a picture the kernel and the image of $T$. Is $T$ invertible? Explain!

**Solution.** Denote by $P$ the orthogonal projection onto the line $x - 2y = 0$ and by $R$ the counterclockwise rotation by $45°$. Then $T = R \circ P$. Let $A$ and $B$ be standard matrices of $P$ and $R$ respectively. Then the standard matrix of $T$ is $BA$. Let us evaluate matrices $A$ and $B$.

Matrix $A$ consists of two columns representing the coordinates of the images of the standard basis vectors $\overline{e}_1$, $\overline{e}_2$ under the transformation $P$:

$$A = \begin{pmatrix} | & | \\ P\overline{e}_1 & P\overline{e}_2 \\ | & | \end{pmatrix}.$$

To evaluate the images we use a formula defining the orthogonal projection $P$:

$$P\overline{x} = \frac{\overline{x} \cdot \overline{u}}{\|\overline{u}\|^2}\overline{u},$$

where $\overline{x}$ is an arbitrary vector in $\mathbb{R}^2$, $\overline{u}$ is a vector along the line of projection, $\|\overline{u}\|$ is its length and $\cdot$ is a dot product. Take $\overline{u} = (2, 1)$. (Note that one can take any vector $\overline{u} = (x, y)$ whose coordinate satisfy the equation of the line $x - 2y = 0$.) Its length is $\|\overline{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$. Calculate the images of $\overline{e}_1 = (1, 0)$ and $\overline{e}_2 = (0, 1)$ under projection $P$:

$$P\overline{e}_1 = \frac{(1,0) \cdot (2,1)}{5}(2,1) = \frac{2}{5}(2,1) = \left(\frac{4}{5}, \frac{2}{5}\right),$$

$$P\overline{e}_2 = \frac{(0,1) \cdot (2,1)}{5}(2,1) = \frac{1}{5}(2,1) = \left(\frac{2}{5}, \frac{1}{5}\right).$$

It gives us the matrix $A$:

$$A = \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

The standard matrix of a counterclockwise rotation by $45°$ is

$$B = \begin{pmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The standard matrix of $T$ is

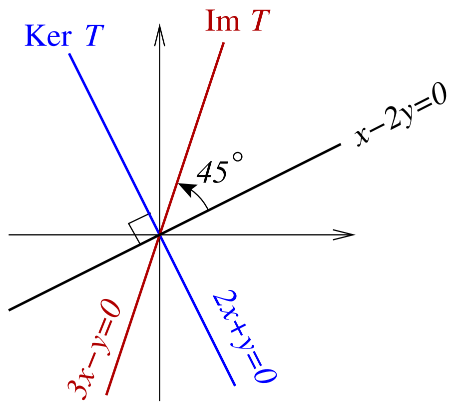
$$BA = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \frac{\sqrt{2}}{10}\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}.$$

The kernel of $T$ is a subspace which is annihilated by $T$. This is a line passing through the origin which is orthogonal to the line $x - 2y = 0$. The equation of this line is $2x + y = 0$. Note that $\operatorname{Ker} T$ is spanned by vector $(1, -2)$ which is annihilated by $T$.

The image of $T$ is the line $x - 2y = 0$ rotated counterclockwise by $45°$ around the origin. The equation of this line is $3x - y = 0$. Note that $\operatorname{Im} T$ is spanned by a column vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ of the matrix of $T$.

Transformation $T$ is not invertible. It can be explained in many different ways. For example, $\operatorname{Ker} T \neq \overline{0}$ or $\operatorname{Im} T \neq \mathbb{R}^2$ or the determinant of the matrix of $T$ is 0.

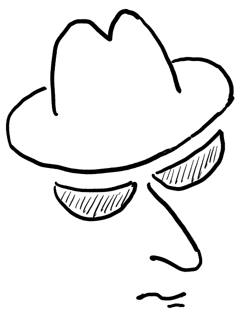
The picture makes all calculations crystal clear:




**Answer:** 

$$\boxed{\begin{array}{l} \text{the standard matrix of } T \text{ is } \dfrac{\sqrt{2}}{10}\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}, \\ \operatorname{Ker} T = \operatorname{span}\{(1,-2)\}, \\ \operatorname{Im} T = \operatorname{span}\{(1,3)\}, \\ T \text{ is } \textbf{not} \text{ invertible.} \end{array}}$$

**Problem 3.**



A secret agent has got an encoded message

$$-4, 2, -19, 0, 3, -9$$

representing the time of the beginning of a secret mission. He knows that the encoding matrix is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & -3 & -3 \end{pmatrix},$$

but nevertheless cannot decode since he is not good in Linear Algebra. Help him to decode the secret time!

**Solution.** Oh, boy! First, invert the matix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{array}\right) \underset{R3-2R1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right) \underset{R3+3R2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 3 & 1 \end{array}\right)$$

$$\underset{\substack{R1-R3 \\ R3\times(-1)}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -3 & -1 \end{array}\right).$$

Second, multiply the inverse matrix by the two vectors $(-4, 2, -19)$ and $(0, 3, -9)$ from the encoded message:

$$\begin{pmatrix} 3 & -3 & -1 \\ 0 & 1 & 0 \\ 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \\ -19 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \qquad \begin{pmatrix} 3 & -3 & -1 \\ 0 & 1 & 0 \\ 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}.$$

Third, read the secret time: $1, 2, 5, 0, 3, 0$ or 12:50:30. (The secret mission is Midterm I : o)

**Answer:** $\boxed{\text{12:50:30}}$

**Problem 4.** For each value of a constant $a$, find the dimension of a subspace generated by vectors $(a, 1, 1)$, $(2, -3, 5)$ and $(1, 0, 1)$.

**Solution.** Let $V = \text{span}\{(a, 1, 1), (2, -3, 5), (1, 0, 1)\}$. The dimension of $V$ is equal to the rank of the matrix

$$\begin{pmatrix} a & 2 & 1 \\ 1 & -3 & 0 \\ 1 & 5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 0 \\ 1 & 5 & 1 \\ a & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 0 \\ 0 & 8 & 1 \\ 0 & 2+3a & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 0 \\ 0 & 8 & 1 \\ 0 & -6+3a & 0 \end{pmatrix}.$$

If $-6 + 3a = 0$, that is $a = 2$, then the rank is 2. If $-6 + 3a \neq 0$, that is $a \neq 2$, then the rank is 3.

**Answer:** $\boxed{\text{If } a = 2 \text{ then the dimension is 2. If } a \neq 2 \text{ then the dimension is 3.}}$

**Problem 5.** A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by

$$T(x, y, z) = (x + y - z, -y + z, -2x - 2y + 2z, 3y - 3z).$$

a) Find the matrix of $T$ with respect to the standard bases.
b) Find a basis in the kernel of $T$ and a basis in the image of $T$.
c) Find the dimensions of the kernel and the image.
d) Find the rank of $T$.
e) Verify the Kernel-Image theorem for $T$.

**Solution.** The matrix of $T$ with respect to the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^4$ is

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -2 & 2 \\ 0 & 3 & -3 \end{pmatrix}_{4\times 3}$$

We perfom elementary row transformations to get the reduced row-echelon form of $A$:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -2 & 2 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A).$$

Obviously, the rank of $A$ is 2. It is the dimension of the image of $T$. The image of $T$ is generated by the first and the second columns of $A$, since the leading ones in the rref($A$) stay in the first and the second columns: $\operatorname{Im} T = \operatorname{span}\{(1, 0, -2, 0), (1, -1, -2, 3)\}$. Since the spanning vectors are linearly independent they comprise a basis of $T$.

The Kernel-Image Theorem says that

$$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim \mathbb{R}^3$$

or

$$\dim \operatorname{Ker} T + 2 = 3.$$

So $\dim \operatorname{Ker} T = 1$. A basis of $\operatorname{Ker} T$ can be found by solving a homogenous linear system with coefficient matrix $A$. It is easy to read the solution from the rref($A$): $x = 0$, $y = t$, $z = t$, where $t$ is an arbitrary real number. Hence

$$\operatorname{Ker} T = \{(0, t, t) \mid t \in \mathbb{R}\} = \operatorname{span}\{(0, 1, 1)\}.$$

**Answer:**

The standard matrix of $T$ is $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -2 & 2 \\ 0 & 3 & -3 \end{pmatrix}$,
a basis of $\operatorname{Ker} T$ is $\{(0, 1, 1)\}$,
a basis of $\operatorname{Im} T$ is $\{(1, 0, -2, 0), (1, -1, -2, 3)\}$,
$\dim \operatorname{Ker} T = 1$,
$\dim \operatorname{Im} T = \operatorname{rk} T = 2$.