MAT 211 Solutions to Midterm 2
Spring 2012.

1. (20 points) Show that the matrix

$$\begin{pmatrix} 1 & 1 & 5 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$$

is invertible and compute the inverse matrix.

Solution: Let us find the inverse matrix using elementary transformations:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Add the first row to the second and the third:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 2 & 7 & 1 & 1 & 0 \\ 0 & 1 & 6 & 1 & 0 & 1 \end{array}\right)$$

Subtract from the second row the third multiplied by 2:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & -2 \\ 0 & 1 & 6 & 1 & 0 & 1 \end{array}\right)$$

Swap the second and third row:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & -2 \end{array}\right)$$

Divide the third row by (-5):

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/5 & -1/5 & 2/5 \end{array}\right)$$

Subtract the third row from the first and the second:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & -1/5 & 6/5 & -7/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 2/5 \end{array}\right)$$

Subtract the second row from the first:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -1/5 & -3/5 \\ 0 & 1 & 0 & -1/5 & 6/5 & -7/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 2/5 \end{array}\right)$$

Therefore

$$A^{-1} = \begin{pmatrix} 1/5 & -1/5 & -3/5 \\ -1/5 & 6/5 & -7/5 \\ 1/5 & -1/5 & 2/5 \end{pmatrix}.$$

2. (20 points). Suppose that the matrices A and B are invertible. Show that the matrix $A \cdot B$ is invertible and its inverse matrix is $B^{-1} \cdot A^{-1}$.

Solution 1: Let us multiply $A \cdot B$ and $B^{-1} \cdot A^{-1}$:

$$(A \cdot B) \cdot (B^{-1} \cdot A^{-1}) = A \cdot (B \cdot B^{-1}) \cdot A^{-1}) = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I.$$

Therefore $A \cdot B$ and $B^{-1} \cdot A^{-1}$ are inverse to each other. Here I denotes the identity matrix.

Solution 2: Suppose that $(A \cdot B)(x) = y$. Then $A(B(x)) = y$, so $B(x) = A^{-1}(y)$ and $x = B^{-1}(A^{-1}(y))$.

The correct examples were graded with partial credit, which actually depended on the complexity of A and B: for example, if $A = B = I$, maximal partial credit was 10 points since the matrices are too trivial

3. a) (10 points) Show that the vectors

$$v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 7 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

form a basis in $\mathbb{R}^4$.

b) (10 points) Compute the coordinates of the vector

$$v = \begin{pmatrix} 4 \\ -1 \\ 7 \\ 4 \end{pmatrix}$$

in this basis.

Solution: Let us solve (b) by transforming the matrix into reduced row-echelon form:

$$\left(\begin{array}{cccc|c} 3 & -1 & 1 & 0 & 4 \\ -1 & 0 & 0 & 0 & -1 \\ 2 & 1 & 3 & 2 & 7 \\ 7 & 2 & -1 & 1 & 4 \end{array}\right)$$

Multiply the second row by (-1):

$$\left(\begin{array}{cccc|c} 3 & -1 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 3 & 2 & 7 \\ 7 & 2 & -1 & 1 & 4 \end{array}\right)$$

Subtract it from all other rows:

$$\left(\begin{array}{cccc|c} 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 2 & 5 \\ 0 & 2 & -1 & 1 & -3 \end{array}\right)$$

Multiply the first row by (-1):

$$\left(\begin{array}{cccc|c} 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 2 & 5 \\ 0 & 2 & -1 & 1 & -3 \end{array}\right)$$

Subtract it from the third and fourth:

$$\left(\begin{array}{cccc|c} 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 2 & 6 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right)$$

Subtract from the third row the fourth multiplied by 4:

$$\left(\begin{array}{cccc|c} 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 10 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right)$$

Divide the third row by (-2) and sort the rows:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

Subtract the fourth row from the third:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

Add the third row to the second:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

We conclude that $v = v_1 + 3v_2 + 4v_3 - 5v_4$. Moreover, since the matrix in the left hand side has rank 4, there are no redundant vectors and v_1, v_2, v_3, v_4 from a basis in $\mathbb{R}^4$.

4. (20 points) Determine which of the following subsets in $\mathbb{R}^n$ are linear subspaces. If they are, compute their dimension.

a) A square on the plane with vertices $(0,0), (5,0), (0,5), (5,5)$.

b) A subset in $\mathbb{R}^3$ defined by the system of equations

$$\begin{cases} 3x - y + 5z & = 0 \\ x + y + z & = 0 \\ x - 3y + 3z & = 0 \end{cases}$$

.

c) A set of all triples (a, b, c) such that the graph of the quadratic polynomial $f(x) = ax^2 + bx + c$ has horizontal tangent at $x = 7$.

Solution: a) No: for example (5,5) belongs to this set by (-5,-5) does not belong to it. One can also say that all possible linear subspaces of plane are 0, a line or the whole plane, and this is none of them.

b) Yes: it is the kernel of the matrix

$$\begin{pmatrix} 3 & -1 & 5 \\ 1 & 1 & 1 \\ 1 & -3 & 3 \end{pmatrix}$$

. If we subtract the second row from the first and the third, we get:

$$\begin{pmatrix} 0 & -4 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 2 \end{pmatrix}$$

. We see that the matrix has rank 2, so the dimension of its kernel equals to 3-2=1: it is a line.

c) Yes: Such a graph has a horizontal tangent at $x = 7$, if $f'(7) = 0$, in other words, $14a + b = 0$. This equation defines the plane in $\mathbb{R}^3$, so its dimension is 2.

5. (20 points) Consider a linear transformation T defined by the matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & 3 & 5 & -7 \\ 0 & -5 & -7 & 5 \end{pmatrix}.$$

a) Compute the basis and dimension of $Ker(T)$.

b) Compute the basis and dimension of $Im(T)$.

c) Check the Rank-Nullity Theorem.

Solution: Let us transform the matrix into reduced row-echelon form. Add first row (multiplied by 2) to the second row:

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 7 & -5 \\ 0 & -5 & -7 & 5 \end{pmatrix}.$$

Add the second row to the third:

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 7 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Divide the second row by 5:

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 7/5 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subtract the second row from the first:

$$T = \begin{pmatrix} 1 & 0 & -2/5 & 2 \\ 0 & 1 & 7/5 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) To find the kernel, we have to solve the system with zero right hand side. We can choose x_3 and x_4 arbitrarily, $x_1 = 2/5x_3 - 2x_4, x_2 = -7/5x_3 + x_4$. If we plug in $x_3 = 1, x_4 = 0$, we get a vector $(2/5, -7/5, 1, 0)$. If we plug in $x_3 = 0, x_4 = 1$, we get a vector $(-2, 1, 0, 1)$. These two vectors form a basis in $Ker(T)$, which is 2-dimensional.

b) To find the image, observe that the third and the fourth columns are redundant, so the basis in the image is formed by the first two columns in T: $(1, -2, 0)$ and $(1, 3, -5)$. The image is also two-dimensional.

c) The rank-nullity theorem says that $\dim Ket(T) + \dim Im(T) = 4$. This equation holds since $\dim Ker(T) = \dim Im(T) = 2$.