MAT211 Lecture 14

- Orthogonal projections and orthogonal basis
- Orthogonality, length, unit vectors
- Orthonormal vectors: definition and properties
- Orthogonal projections: definition, formula and properties.
- Orthogonal complements
- Pythagorean theorem, Cauchy inequality, angle between two vectors

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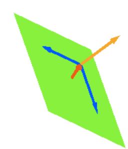
- Two vectors u and v in R^n are perpendicular or orthogonal if u.v=0
- The length of a vector v in R^n is $||v||=\sqrt{v.v}$
- A vector v in R^n is called a unit vector if $||v||=1$

Example
- Find a unit vector in the line spanned by (1,1,3)
- Find a vector of length 2 orthogonal to (1,1,3)

- A vector v in R^n is orthogonal to a subspace V of R^n if it is orthogonal to all vectors in V
- If $(b_1,b_2,..,b_m)$ is a basis of V, then v is orthogonal to V if (and only if) v is orthogonal to $b_1,b_2,..$ and b_m.

Example
- Consider the subspace V of R^3 span by (1,1,1) and (1,0,1).
- Find all the vectors orthogonal to V.



The vectors $u_1, u_2, .. u_m$ of R^n are called orthonormal if they are all unit vectors and are orthogonal to one another. In symbols

$$(u_i,u_j)=0 \text{ if } i \neq j$$
$$(u_i,u_i)=1$$

Example: Find an orthonormal basis of the subspace of R^3 of equation x+y+z=0.

- Orthonormal vectors are linearly independent.
- A set of n orthonormal vectors in R^n form a basis.

Consider the vectors $v_1=(1/\sqrt{2})(1,0,1)$,
$V_2=(0,1,0)$, $v_3=(1/\sqrt{2})(1,0,-1,)$.

- Check that v_1, v_2 and v_3 and orthonormal.
- Are they linearly independent?

- Let V be a subspace of R^n and let x be a vector R^n. Then there exists unique vectors $x^{\perp}$ and $x^{||}$ such that
- $x = x^{||} + x^{\perp}$
- $x^{||}$ in V
- $x^{\perp}$ is orthogonal to V.
- If V us a subspace of R^n with orthonormal basis $(b_1,b_2,..,b_m)$ then

$$\mathrm{proj}_V\,(x)=(b_1.x)\,b_{1+}\,(b_2.x)\,b_{2+\dots}\,(b_m.x)\,b_m$$

- In particular if V= R^n

$$x=(b_1.x)\,b_{1+}\,(b_2.x)\,b_{2+\dots}\,(b_n.x)\,b_n$$

- Find the orthogonal projection of (1,2,3) onto the subspace of R^3 of equation x+y+z=0.
- Write (1,2,3) as a linear combination of the vectors $v_1=(1/\sqrt{2})(1,0,1)$, $v_2=(0,1,0)$, $v_3=(1/\sqrt{2})(1,0,-1,)$.

- Consider V a subspace of R^n. The <u>orthogonal complement $V^{\perp}$ of V</u> is the set of vectors x of R^n that are orthogonal to all vectors in V.
- In other words $V^{\perp}$ is the kernel of the linear transformation proj_V

If V is a subspace of R^n

- The orthogonal complement of V is a subspace of R^n
- $V \cap V^{\perp} =\{0\}$
- $\dim(V)+\dim(V^{\perp})=n$
- $(V^{\perp})^{\perp}=V$

- Find the orthogonal complement V where V is the subspace of R^3 of equation x+y+z=0.

Theorem: Consider two vectors x and y in R^n

- $||x+y||^2=||x||^2 +||y||^2$ if an only if x and y are orthogonal (Pythagorean theorem)
- If V is a subspace of R^n then $||\mathrm{proj}_V(x)||\leq||x||$
- Cauchy-Schwarz Inequality: $|x.y|\leq||x||.||y||$.

- Consider two non-zero vectors x and y in R^n. The angle θ between these two vectors is defined as arc $\cos(x.y/||x||.||y||)$.

- Find the angle between the vectors x=(1,1,1) and (1,0,1).