MY NAME IS: SOLAR ID:

Problem	1	2	3	4	Total	Extra Credit
Score						
Total Score	25	25	25	25	100	20

MAT 211 - Introduction to linear algebra, Midterm 1

Oct 14th, 2009

SHOW ALL WORK TO GET FULL CREDIT; A CORRECT ANSWER WITH INCORRECT OR NO JUSTIFICATION **will not get credit**.

CROSS OUT THE WORK YOU DO NOT WANT TO BE GRADED.

(1) Consider the the subset W of $R^{3\times 3}$ formed by all diagonal 3×3 matrices.

(a) (12 points) Is W a subspace of $R^{3\times 3}$?

(b) (13 points) Find a basis of W.

a) Yes, it is. Justification:

1) W is closed under addition

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} + \begin{pmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix} = \begin{pmatrix} a+e & 0 & 0 \\ 0 & b+f & 0 \\ 0 & 0 & c+g \end{pmatrix} \text{ in } W$$

2) W is closed under scalar multiplication

$$\lambda \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} \lambda a & 0 & 0 \\ 0 & \lambda b & 0 \\ 0 & 0 & \lambda c \end{pmatrix} \text{ in } W$$

3) 0 is in W, $0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is diagonal,

b) A basis is $B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

(2) Consider the transformation T from P_2 to P_2 defined by $T(f(t)) = 3f'(t)$.

(a) (7 points) Determine whether T is linear

(b) (8 points) Determine the kernel and image of T.

(c) (4 points) Find rank and nullity of T.

(d) (6 points) Is it an isomorphism?

a) . T is closed under addition

If f and g are in P_2 then $T(f+g) = 3(f+g)'$

$= 3f' + g'$

$= T(f) + T(g)$

. T is closed under scaler multiplication

If f in P_2 and λ is in $\mathbb{R}$, $T(\lambda f) = 3(\lambda f)' = \lambda 3f' = \lambda T(f)$

b) An element f in P_2 can be written as $at^2 + bt + c$.

f is in Ker T iff $T(at^2 + bt + c) = 0$

$\overset{\shortparallel}{3at + b}$. That is, $3at+b$ is the zero polynomial. This implies $a=0$ and $b=0$

Therefore $\boxed{\text{Ker } T = \{ f \text{ in } P^2 \,/\, f(t) = c \text{ for all } t \}}$

An element f is in Im T if $f(t) = T(at^2 + bt + c)$ for some a, b, c in $\mathbb{R}$

Thus $f(t) = 3at + b$, Hence $\text{Im } T = \{ f \text{ in } P_2 \,/\, f(t) = et + f \}$

c) A basis of Ker T is $\{1\}$. ~~The~~ Since Ker T has a basis with one element, then nullity is 1.

A basis of Im T is $\{1, t\}$. Thus rank is two.

d) Since Ker T contains vectors ~~a~~ which are not the zero vector then ~~the~~ T is not an isomorphism

(3) Recall that $U^{2\times 2}$ is the space of 2×2 upper triangular matrices. Consider the linear transformation T from U to U defined by $T(M) = MA$ where A is the matrix $\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$

Let $\mathbf{A}$ and $\mathbf{B}$ be two basis of U given by $\mathbf{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$ and $\mathbf{A} = \left(\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.\right)$

(a) (6 points) Find the matrix of T with respect to the basis $\mathbf{B}$.

(b) (7 points) Find the change of basis matrix from $\mathbf{B}$ to $\mathbf{A}$.

(c) (6 points) $\boxed{\text{Not easy!}}$ Consider a matrix M in $U^{2\times 2}$ such that $[M]_{\mathbf{A}} = [1, 0, 1]$. Find M and $[M]_{\mathbf{B}}$.

(d) (6 points) Give another basis of $U^{2\times 2}$ (different from $\mathbf{B}$ and different from $\mathbf{A}$).

(a). $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\left[T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_B = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\left[T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]_B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \left[T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]_B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So the matrix shold be $\begin{pmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$

b) $\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_A = -2\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + (1)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]_A = -1\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + 1.\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]_A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$S_{B\to A} = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

c) $M = 1.\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $[M]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

d) see at the end.

(4) Let $W = span\{(1,1,-1,1),(1,2,0,0)\}$ in R^4.

(a) (12 points) Use Gram-Schimdt to find an orthonormal basis of W.

(b) (13 points) Find the orthogonal projection of $(1,0,0,\frac{2}{5})$ onto W.

(a) $P_1 = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$ $\quad P_2 = V_2 - \dfrac{P_1 \cdot V_2}{P_1 \cdot P_1} P_1 = \begin{pmatrix}1\\2\\0\\0\end{pmatrix} - \dfrac{\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}\begin{pmatrix}1\\2\\0\\0\end{pmatrix}}{\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}} \begin{pmatrix}1\\1\\-1\\1\end{pmatrix}$

$$= \begin{pmatrix}1\\2\\0\\0\end{pmatrix} - \frac{3}{4}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1/4\\5/4\\3/4\\-3/4\end{pmatrix}$$

. $\{P_1, P_2\}$ is an orthogonal basis

. $O_1 = \dfrac{P_1}{\|P_1\|} = \begin{pmatrix}1/2\\1/2\\-1/2\\1/2\end{pmatrix}$ $\quad O_2 = \dfrac{P_2}{\|P_2\|} = \dfrac{\sqrt{4}}{\sqrt{11}}\begin{pmatrix}1/4\\5/4\\3/4\\-3/4\end{pmatrix} = \dfrac{1}{\sqrt{11}}\begin{pmatrix}1/2\\5/2\\3/2\\-3/2\end{pmatrix}$

$\boxed{\{O_1, O_2\} \text{ is an orthonormal basis}}$

(b) $P_W\begin{pmatrix}1\\0\\0\\2/5\end{pmatrix} = \left(O_1 \cdot \begin{pmatrix}1\\0\\0\\2/5\end{pmatrix}\right) \cdot O_1 + \left(O_2 \cdot \begin{pmatrix}1\\0\\0\\2/5\end{pmatrix}\right) O_2$

$$= \left(\frac{1}{2} + \frac{2}{10}\right)\begin{pmatrix}1/2\\-1/2\\1/2\\1/2\end{pmatrix} + \frac{1}{\sqrt{11}}\left(1/2 - \frac{6}{10}\right)\frac{1}{\sqrt{11}}\begin{pmatrix}1/2\\5/2\\3/2\\-3/2\end{pmatrix}$$

(5) $\boxed{\text{Extra credit}}$

(a) (5 points) Give an example of a linear transformation T between two linear spaces V and W such that $\mathrm{im}T = W$ but T is not an isomorphism.

(b) (5 points) Give an example of a linear transformation T between two linear spaces V and W such that $\mathrm{Ker}T = \{0\}$ but T is not an isomorphism.

(c) (5 points) Find the orthogonal complement of W, $W^\perp$, where W is the subspace of problem 4 .

(d) (5 points) If T is the linear transformation of Problem 2, find the matrix of T with respect to the basis $(4, t-4, (t-4)^2)$

a) T from $\mathbb{R}^2$ to $\mathbb{R}$, $T(x,y) = x$

b) T from $\mathbb{R}$ to $\mathbb{R}^2$ $T(x) = (x,x)$

c) $W^\perp = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle/ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = 0 \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle/ x_1 + x_2 - x_3 + x_4 = 0 \text{ and } x_1 + 2x_2 = 0 \right\}$

d) $[T(4)]_B = [0]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[T(t-4)]_B = [3]_B = \begin{pmatrix} 3/4 \\ 0 \\ 0 \end{pmatrix}$

$[T((t-4)^2)]_B = [6(t-4)]_B = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$ Then the matrix is $\begin{pmatrix} 0 & 3/4 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

3) d) $\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$