MAT211 Lecture 12

Linear Transformations and isomorphisms

- Linear transformations, image, rank, nullity
- Isomorphism and isomorphic spaces
- Theorem: Coordinate transformations are isomorphisms
- Properties of isomorphisms

Definition

- Consider two linear spaces V and W.
- A function T from V to W is called a linear function if for every pair of elements f and g in V, and every scalar k,
- T(f + g) = T(f)+T(g)
- T(k.f) = k. T(f)

EXAMPLE: Find out whether the following transformations from R^{2X2} to R^{2X2} are linear.

- $T(M)=M^2$
- T(M)=7M
- $T(M)=P M P^{-1}$ where P is $\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$

EXAMPLE: Find out whether the following transformation from R^{2X2} to R^3 is linear.

- T(M)=(a,b,0) where M is $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Definition

- The image of a linear transformation T from V to W, denoted by Im T, is the subset of W {T(f) : f in V}.
- The kernel of a linear transformation T from V to W, denoted by ker T, is the subset of V {f in V : T(f)=0}.

Definiton

- If the image of a linear transformation T is finite dimensional, then the dimension of im T is called the *rank* of T.
- If the kernel of a linear transformation T is finite dimensional then the dimension of kernel of T is called *nullity* of T.

EXAMPLE: Find rank, image, kernel and nullity of the following transformation from R^{2X2} to R^3.

- T(M)=(a,b,0) where M is $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

EXAMPLE

- Find kernel and nullity of the transformations of T, T(M)=MA-AM, where A is the matrix $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

EXAMPLE

- Find out if the transformations from P_2 to R defined by $T(f(t)) = \int_{-2}^{3} P(t)dt$ is linear.
- Find image, rank, kernel and nullity.

Definition

- An invertible linear transformation is called an *isomorphism*.
- Two linear spaces V and W are *isomorphic* if there exists an isomorphism T from V to W.

Theorem

- A linear transformation T from V to W is an isomorphism if and only if ker(T)={0} and im(T)=W.

EXAMPLE

- Is the transformation T(M)=MA-AM from R^{2x2} to R^{2x2} an isomorphism? A is the matrix $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

EXAMPLE

- Find out if the transformations from P_2 to R defined by $T(f(t)) = \int_{-2}^{3} P(t)dt$ is linear.
- Find image, rank, kernel and nullity.
- Is it an isomorphism?

EXAMPLE

- Consider the linear transformation T from P_2 to P_1 given by $T(p(t))=p'(t)$.
- Is it an isomorphism?
- Find rank, nullity, image and kernel.

Theorem

If $B=(f_1, f_2,\ldots,f_n)$ is a basis of a vector space V, then the coordinate transformation $L_B(f)=[f]_B$ from V to R is an isomorphism.

Hence, any n-dimensional vector space is isomorphic to R^n.

EXAMPLE

Find a basis of the linear space of 2 x 2 matrices annd find the coordinate transformation for that basis.

Theorem: If V and W are finite dimensional linear spaces then

- If V is isomorphic to W then dim(V)=dim(W).
- If T is a linear transformation from V to W and ker(T)=0, and dim(V)=dim(W) then T is an isomorphism.
- If T is a linear transformation from V to W and im(T)=W, and dim(V)=dim(W) then T is an isomorphism.