MAT211 Lecture 11

- Definition Linear Spaces
- Examples
- Subspaces
- Span, linear independence, basis, coordinates
- Coordinate transformation
- Dimension
- Differential equation - Solution space

Definition

A *linear (or vector) space* V is a set of elements endowed with two operations

- \+ addition: for each f and g in V, f+g is an element in V.
- . multiplication: For each f in V and each k in R, k.v is an element in V.

Moreover, these operation satisfy the following properties:

Definition (cont) for each f and g in V

(f + g) + h = f + (g + h)

f + g = g + f

There exists a unique element in V, denoted by **0** and called the neutral element such that f + **0** = **0** + f = f

For each f in V there exists a unique element in V denoted by -f such that f+(-f)=0.

Definition (cont) for each f and g in V, each c and k in R,

k. (f + g) = k.f + k.g

(c + k). f = c. f + k.f

c.(k.f) = (c.k).f

1.f=f

EXAMPLES of Linear Spaces:

- R^n.
- The set of all m x n matrices
- The space of 2 x 2 matrices such that a+d=0 $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- The set of all polynomials.

Questions: What is +, . , **0** , if v is in the linear space, what is -v?

MORE EXAMPLES of Linear Spaces:

- The set of all infinite sequences of real numbers. (addition and mult. are defined term by term)
- F(R,R) the set of all functions from R to R.
- The set of all polynomials of degree n at most n.
- Set of geometric vectors in plane.

Question: What is +, ., **0**, if v is in the linear space, what is -v?

Definition

We say that an element f of a linear space V is a *linear combination* of the elements $f_1, f_2, \ldots, f_n$ of V if there exists scalars such that

$f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$

EXAMPLE

If V is the space of 2 x 2 matrices $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ such that a+d=0, show that the

$\begin{vmatrix} 2 & -2 \\ -2 & -2 \end{vmatrix}$ is a linear combination of $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ and $\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix}$

EXAMPLE

Is the polynomial $x^2 + x + 1$ a linear combination of $x^2 + 1$, $x^2 - 1$ and $3x+3$?

Definition

A subset W of a linear space V is a subspace if

- W contains the neutral element of V.
- If f and g are in W, so is f+g.
- If f is in W and k is a scalar then k.f is in W.

EXAMPLE

Let V be the space all of 2 x 2 matrices Show that subset W of all

the matrices $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

such that a+d=0 is a subspace of V.

EXAMPLE:

Show that the subset of F(R,R) of all functions such that f(0)=0 is a subspace of F(R,R)

EXAMPLE

- Is the set of all 2 x 2 invertible matrices a subspace of the linear space formed by all 2 x 2 matrices?
- Denote by P_n the set of all polynomials of degree at most n. Is P_2 a subspace of P_n?
- Is the subset of all polynomials of degree 2 a subset of P_n?

Definition: Consider the elements f_1, $f_2,\dots,f_n$ in V

- $f_1, f_2,\dots,f_n$ *span V* if every element in V is a linear combination of the elements $f_1, f_2,\dots,f_n$.
- f_i is *redundant* if it a linear combination of t f_1, $f_2,\dots,f_{i-1}$
- $f_1, f_2,\dots,f_n$ are *linearly independent* if none of them is redundant.
- $f_1, f_2,\dots,f_n$ form a *basis* if they are linearly independent and span V.

Example

Consider the linear space M of all matrices 2x3.

Find a finite set that span M

Find a basis of M.

Can you find a basis that does not span?

Can you find a subset that span but it is not a basis? If so, indicate the redundant vectors.

Suppose that the elements $f_1, f_2,\dots,f_n$ are a basis of a vector space V.

Then any element f in V can be written as $c_1 f_1 + c_2 f_2 + \dots + c_n f_n$ for some scalars $c_1, c_2, \dots, c_n$

The coefficients $c_1, c_2, \dots, c_n$ are called the *coordinates of f with respect to the basis B*= $(f_1, f_2,\dots,f_n)$

The vector $[c_1, c_2, \dots, c_n]$ in R^n is called the *coordinate vector* of f and denoted by $[f]_B$

EXAMPLE

Find a basis B of P_2. Find the coordinates of the polynomial (x-1)(x+1) with respect to B.

Suppose that the elements $f_1, f_2,\dots,f_n$ are a basis of a vector space V.

The vector $[c_1, c_2, \dots, c_n]$ in R^n is called the *coordinate vector* of f and denoted by $[f]_B$

The transformation L: V -> R^n, defined by L(f)=[f] is called the B-coordinate transformation.

If f and g are in V and k is a scalar then

L(f+g)=L(f)+L(g) and L(k.f)=kL(f).

EXAMPLE

For the basis of P_2 we find in the previous example, describe the transformation

$L: P_2 \to R^n$.

If $f=(x-1)(x+1)$ and $g=x^2 -3x+ \pi$, check that

- $L(f+g)=L(f)+L(g)$
- $L(k.f)=k.L(f)$.

Theorem

If a basis of a vector space has n elements then all basis of a linear space have the n elements.

Definition: If a linear space V has a basis with n elements we say that the dimension of V is n.

EXAMPLE: Find a basis and determine the dimension:

- The linear space of all 2 x 2 matrices.
- The space of all 2 x 2 matrices such that a+d=0.
- P_2 , the space of all polynomials of degree at most 2.
- The space of all 2 x 2 matrices that commute with $\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$

EXAMPLE: Find a basis and determine the dimension:

- The space of all polynomials.

Theorem

If a basis of a vector space has n elements then all basis of a linear space have the n elements.

Definition: If a linear space V has a basis with n elements we say that the dimension of V is n, and that V is finite dimensional.

EXAMPLE

Find a basis of the linear space W of all

the matrices

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

such that a+d=0.