MAT 132 Review

Chapter 5

In each of the cases below, decide which method apply. Evaluate 32.

23. $\int_0^5 \frac{x}{x+10}\,dx$

24. $\int_0^5 ye^{-0.6y}\,dy$

25. $\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2+\cos t}\,dt$

26. $\int_1^4 \frac{dt}{(2t+1)^3}$

27. $\int_1^4 x^{3/2} \ln x\,dx$

28. $\int \sin x \cos(\cos x)\,dx$

29. $\int \frac{dt}{t^2+6t+8}$

30. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

31. $\int e^{\sqrt[3]{x}}\,dx$

32. $\int \tan^{-1}x\,dx$

55–60 Evaluate the integral or show that it is divergent.

55. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

56. $\int_0^\infty \frac{\ln x}{x^4}\,dx$

57. $\int_{-\infty}^0 e^{-2x}\,dx$

58. $\int_0^1 \frac{1}{2-3x}\,dx$

59. $\int_1^e \frac{dx}{x\sqrt{\ln x}}$

60. $\int_2^6 \frac{y}{\sqrt{y-2}}\,dy$

Hints

Chapter 5

23. $\int_0^5 \frac{x}{x+10}\,dx$ (x+10-10)/(x+10)

24. $\int_0^5 ye^{-0.6y}\,dy$ parts

25. $\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2+\cos t}\,dt$ =0, odd function f(-x)=-f(x)

26. $\int_1^4 \frac{dt}{(2t+1)^3}$ change of variable u=2t+1

27. $\int_1^4 x^{3/2} \ln x\,dx$ parts, "remove ln"

28. $\int \sin x \cos(\cos x)\,dx$ change of variable cos(x)=u

29. $\int \frac{dt}{t^2+6t+8}$ partial fractions

30. $\int \frac{x}{\sqrt{1-x^4}}\,dx$ change of variable u=x^2

31. $\int e^{\sqrt[3]{x}}\,dx$ parts, twice

32. $\int \tan^{-1}x\,dx$

55–60 Evaluate the integral or show that it is divergent.

55. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

56. $\int_0^\infty \frac{\ln x}{x^4}\,dx$

57. $\int_{-\infty}^0 e^{-2x}\,dx$

58. $\int_0^1 \frac{1}{2-3x}\,dx$

59. $\int_1^e \frac{dx}{x\sqrt{\ln x}}$

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Table of Indefinite Integrals

$$\int k\,dx = kx + C$$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{a^x}{\ln a} + C$ $a \neq 1$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$

$\int \csc x \cot x\,dx = -\csc x + C$

$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

Techniques of integration

- Properties of functions (Odd, even)
- Properties of integrals
  - Example: ∫(f(x)+g(x))dx=∫f(x)dx+∫g(x)dx **(trial an error)**
- Tables of integrals **(always works)**
- Rewrite
  - Example: Partial fractions **(always works)**
  - Example: trigonometric identities **(trial an error)**
- Parts: **(trial an error)**
- Substitution **(trial an error)**
  - Example: Trigonometric substitution.

Techniques of integration

- Properties of functions (Odd, even)
- Rewrite
  - Partial fractions: When there is a quotient of polynomials.
  - Trigonometric identities ($\cos^2 x + \sin^2 x = 1$, cos and sin of double angle): Example: to find $\int \cos^n x \, dx$ (n odd or even)
- Parts: (to "remove" integer powers of x or functions which do not appear as integrand in the table) Example $\int \ln(x)dx$, $\int xe^x dx$, $\int x\cos(x) dx$.
- Substitution
  - Example: Trigonometric substitution. The integrand contains an expression of the form $(a^2-x^2)^{1/2}$, $(a^2+x^2)^{1/2}$, $(x^2-a^2)^{1/2}$ where a is a positive number.

Chapter 6

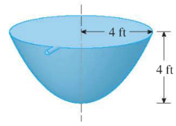
6. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^{-2x}$, $y = 1 + x$, and $x = 1$.

21. The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an equilateral triangle with side $\frac{1}{4}x$ meters. Find the volume of the monument.

29. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.
(a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.

(use that water weights 62.5 lb/ft^3)

(b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?



Chapter 6

6. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^{-2x}$, $y = 1 + x$, and $x = 1$.

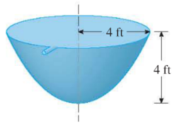
$\frac{25}{12}\pi + \frac{1}{4}\pi e^{-4}$

Recall disk, washer, shells

29. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.
(a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.

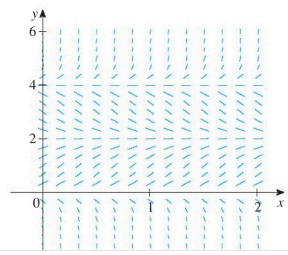
(b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?

In work by a variable force problems, recall "divide and conquer"



(a) A direction field for the differential equation $y' = y(y-2)(y-4)$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.
(i) $y(0) = -0.3$ (ii) $y(0) = 1$
(iii) $y(0) = 3$ (iv) $y(0) = 4.3$

(b) If the initial condition is $y(0) = c$, for what values of c is $\lim_{t\to\infty} y(t)$ finite? What are the equilibrium solutions?



Chapter 7, 8

5–6 Solve the differential equation. and solve the IVP with y(0)=1

5. $2ye^{y^2}y' = 2x + 3\sqrt{x}$

20. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?

Mixing problems key point:
Q(t)=amount of salt (or other) in the tank at time t.
Model rate of change of Q(t)

Examples to remember:

1 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

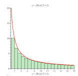
Series Strategies.

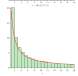
- If $\{a_n\}$ does not converge to 0 when $n \to \infty$, then $\sum a_n$ diverges.
- Recall convergence of p-series and geometric series.
- If the series has a form that is similar to a p-series or a geometric series, then try the comparison tests. (Recall that the comparison tests apply only to series with positive terms.)
- Series that involve factorials or other products (including a constant raised to the n-th power) are often conveniently tested using the ratio test. (Note the ratio test does not work with the p-series)
- If the series is alternating, $a_n > 0$, $a_n \le a_{n+1}$, and $a_n \to 0$ wher $\sum_{n=1}^{\infty} (-1)^n a_n$ $n \to \infty$)
- If $a_n = f(n)$, $f(x) > 0$ for all $x \ge 1$, f is continuous, and decreasing and the integral $\int f(x)\,dx$ is easily evaluated, then try the integral test.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

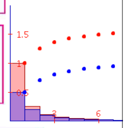




The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(a) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(b) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.



The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

1 Theorem If a series $\sum a_n$ is absolutely convergent, then it is convergent.

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i) } b_{n+1} \le b_n \qquad \text{and} \qquad \text{(ii) } \lim_{n\to\infty} b_n = 0$$

then $$|R_n| = |s - s_n| \le b_{n+1}$$

3 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \ge n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$

9 Taylor's Inequality If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d$$

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

- The number R in (iii) is called the <u>radius of convergence</u> of the power series.
- By convention, R=0 in (i) and R=∞ in (ii)
- The <u>interval of convergenc</u>e is the set of all values of x for which the power series converges.
- There are four possibilities for a power series centered at a.
  - (a-R,a+R)
  - [a-R,a+R)
  - (a-R,a+R]
  - [a-R,a+R]

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

21. The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an equilateral triangle with side $\frac{1}{4}x$ meters. Find the volume of the monument.

9–18 Determine whether the series is convergent or divergent.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

14. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

Chapter 8

23. Express the repeating decimal 1.2345345345 . . . as a fraction.

25. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ correct to four decimal places.

26. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/n^6$ and estimate the error in using it as an approximation to the sum of the series

30–33 Find the radius of convergence and interval of convergence of the series.

30. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\,4^n}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$

34. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

37–44 Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (definition of a Maclaurin series) or known series such as geometric series, binomial series, or the Maclaurin series for e^x, $\sin x$, and $\tan^{-1}x$.

37. $f(x) = \dfrac{x^2}{1+x}$

38. $f(x) = \tan^{-1}(x^2)$

39. $f(x) = \ln(4-x)$

40. $f(x) = xe^{2x}$

41. $f(x) = \sin(x^4)$

42. $f(x) = 10^x$

46. Use series to approximate $\int_0^1 \sqrt{1+x^4}\,dx$ correct to two decimal places.

49. Use series to evaluate the following limit.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$