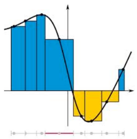
MAT 132 Lecture 1: Review.

- Limits and Derivatives: Incredibly fast review
- Integrals: Review (also a bit fast)



Example from "A tour of the Calculus" By D. Berlinksi

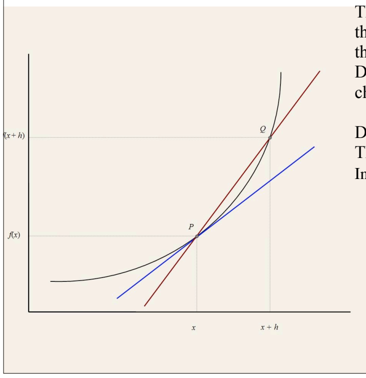
- Crossing room, Zeno affirmed that a man must first cross half of the room, then half of the half that remains, and then half of the half that then remains, "This process" wrote Zeno, "can always be continued and can never be ended." But infinite process requires an infinite amount of time for its completion, no?
- How can this be explained?

The concept of LIMIT

Can you write down the exact value of √2?

- 1
- 1.4
- 1.41
- 1.414
- 1.4142
- 1.41421
- ....

What about derivatives?



The closer point Q gets to point P, the closer the secant slope gets to the tangent slope.
Derivatives give us the "rate of change" of a function.

Demonstrations:
The Tangent Line Problem
Instantaneous Rate of Change

what is this magnifying glass doing here?

Derivatives: Some formulae and properties (taken from the cyberspace)

Derivatives

(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$;

Particularly, we note that

$\frac{d}{dx}(x)=1$;

(ii) $\frac{d}{dx}(\sin x)=\cos x$;

(iii) $\frac{d}{dx}(-\cos x)=\sin x$;

(iv) $\frac{d}{dx}(\tan x)=\sec^2 x$;

(v) $\frac{d}{dx}(-\cot x)=\mathrm{cosec}^2 x$;

(vi) $\frac{d}{dx}(\sec x)=\sec x\tan x$;

(vii) $\frac{d}{dx}(-\mathrm{cosec}\, x)=\mathrm{cosec}\, x\cot x$;

(viii) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$;

(ix) $\frac{d}{dx}(-\cos^{-1}x)=\frac{1}{\sqrt{1-x^2}}$;

(x) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$;

(xi) $\frac{d}{dx}(-\cot^{-1}x)=\frac{1}{1+x^2}$;

(xii) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$;

(xiii) $\frac{d}{dx}(-\mathrm{cosec}^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$;

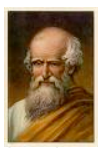
(xiv) $\frac{d}{dx}(e^x)=e^x$;

(xv) $\frac{d}{dx}(\log|x|)=\frac{1}{x}$;

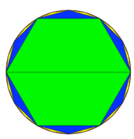
(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right)=a^x$;

1. $(cf)'=cf'(x)$
2. $(f\pm g)'=f'(x)\pm g'(x)$
3. $(fg)'=f'g+fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c)=0$
6. $\frac{d}{dx}(x^n)=nx^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x)))=f'(g(x))g'(x)$
   This is the **Chain Rule**

Integration: (One of the) precursors



Arquimedes
287 - 212 B.C.E.

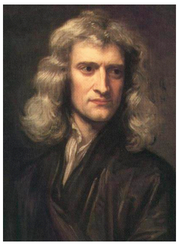


Approximating Pi (the area of a circle with radius 1)

Mathematica Demonstration

Integration: The creators

Gottfried Wilhelm Leibniz
1646 - 1716



Sir Isaac Newton
1642 - 1727

- During the semester, I will send a few emails through Blackboard. Please make sure that your email address is updated.
- This is a large class, so from now on, there are certain email messages that I will not be able to answer, for instance:
  - Messages whose answer is contained in the course website
  - Asking something that you could have asked to a classmate (like "what did you cover on Monday?")
  - Messages telling me something you could have told me the following day in class.
- I will answer messages about appointments to discuss my favorite subject, Math (remember to include possible meeting times) or course related question that really require a timely answer.

Indefinite Integral ∫f(x)dx

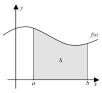
- ***antiderivative*** of a function (An antiderivative of a function f is another function whose derivative is f)
- The "answer" is another function, or better said, a family of functions

Don't forget the constant

- There are no limits of integration
- Example: Find ∫f(x)dx if f(x)=2x

Definite Integral $\int_1^2 f(x)dx$

- When the function f is positive, it is the area between the x-axis and the graph of the function f.



- The "answer" is a number
- The numbers 1 and 2 are the "limits of integration".
- Example if f(x)=2x, what is

$$\int_1^2 f(x)dx$$

Both types of integrals exist only if the function "f" is "good enough". All the functions we are going to study are "good

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Definite Integral

- When the function f is positive, it is the area between the x-axis and the graph of the function f.
- In general is the signed area.

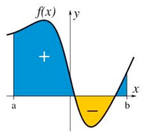
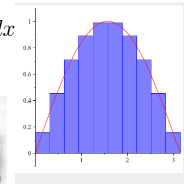


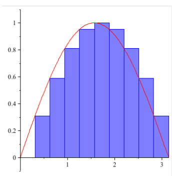
Image from Wikipedia

More about the definite integral

Find the difference between the two Riemann sums.

- Suppose we want to estimate $\int_0^{\pi} \sin(x)dx$
- The definite integral is the limit of the Riemann Sums,
- Now, what is a Riemann sum???





Example :
1. Use the midpoint rule to estimate the integral above. Divide [0,π] into 3 intervals.
2. Use the left-hand rule with n=4.
3. Maple Demonstration

Properties of the definite integral

1: If $a = b$, th $\int_a^a f(x)\,dx = 0$

2: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

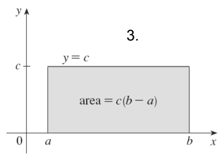
3: $\int_a^b c\,dx = c(b-a)$, where c is any constant

4: $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

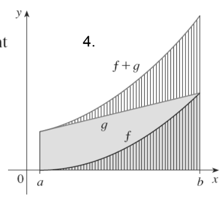
5: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

6: $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

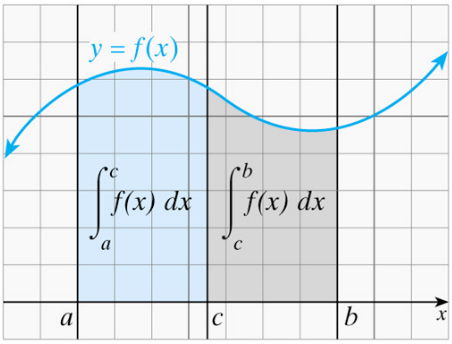
3.



4.



7: $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

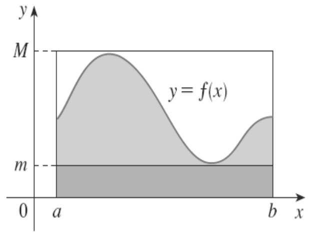


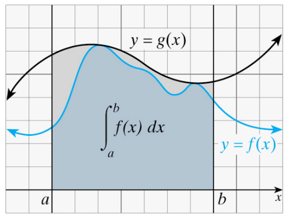
8: If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

9: If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

10: If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

11: If f is continuous on $[a, b]$, than $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$





The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

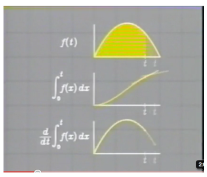
1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

The Fundamental Theorem of Calculus is indeed, FUNDAMENTAL.

It gives a
"recipe" to compute definite integrals.

Otherwise, we would have to spent our lives calculating Riemann sums, rectangle by rectangle.



Fundamental Theorem Of Calculus

http://www.youtube.com/watch?v=gMdh_fiGZag

Maple

The point of the Fundamental theorem is to take something you want to know, and translate it into something you have hope of calculating.

Correct!

EXAMPLE

Let $g(x) = \int_0^x \sin(t)dt,\ x \in [0, \pi]$.
Using only the definition of integral as the signed area under the graph.

- Evaluate $g(0)$ and $g(\pi)$.
- On what interval g is increasing?
- Does g have a maximum value?
- Sketch a graph of g.
- Sketch a graph of g'

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Table of Indefinite Integrals

$$\int k\,dx = kx + C$$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$ | $\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$ | $\int a^x\,dx = \frac{a^x}{\ln a} + C$ a ≠1

$\int \sin x\,dx = -\cos x + C$ | $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$ | $\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$ | $\int \csc x \cot x\,dx = -\csc x + C$

$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$ | $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

An insane mathematician gets on a bus and starts threatening everybody: **"I'll integrate you! I'll differentiate you!!!"** Everybody gets scared and runs away. Only one lady stays. The guy comes up to her and says: **"Aren't you scared, I'll integrate you, I'll differentiate you!!!"**

The lady calmly answers: **"No, I am not scared, I am e^x ."**

- A little boy is on top of the Empire State Building and accidentally drops his teddy bear. His parents panic because he cannot sleep without his teddy bear. They can get to the street in 9 seconds. Can they reach the teddy bear before somebody else find it?
- (recall gravity imparts an acceleration of -32ft/sec^2 to any object falling near the surface of the Earth.)
- Height Empire State Building:1,250 feet

EXAMPLE

- (5.3 - 18) Evaluate the integral

$$\int_0^5 (2e^x + 4\cos x)dx$$

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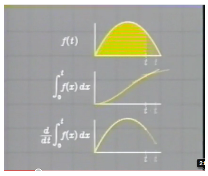


EXAMPLE

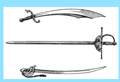
- (5.3 - 46) Evaluate the integral

$$\int \sec t(\sec t + cost)dt$$

From "How to ace calculus"

As you can imagine, it is a rare integral indeed that willingly allows itself to be integrated. Most fight with claw and threaten lawsuits. If we want to successfully trap and tame the wild integrals and teach them proper table manners, we need a variety of snares.

Here are some of the "weapons" we will use to tame wild integrals (that is, techniques to find antiderivatives)
Substitution
Integration by parts.
Trigonometric sustituton
Partial fractions



Disclaimer: we will not be able to find antiderivatives for all functions we know

Integration by Substitution

- Some integrals cannot be easily solved by just applying the previous rules of integration. For instance,

$$\int x^2 \cos(x^3 - 2)dx$$

$$\int \frac{1}{3}\cos(u)du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(x^3 - 2) + C$$

Recall the Chain Rule for derivation

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

In the previous example

f(x)=cos(x)

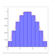
g(x)=x^3-2

$$\int x^2 \cos(x^3 - 2)dx$$

HINT: If the function to be integrated (cos(x^3-2)) contains a "piece" (x^3-2) whose derivative "is" a factor of the function (x^2), then that piece is a candidate for substitution.

Always remember to go back to the original variable.

We have reviewed

- Definite integral (signed area)
- Indefinite integral (all the antiderivatives)
- Riemann's sums 
- Properties of integrals.
- Table of integrals of some functions.
- Fundamental Theorem of Calculus (thus we can evaluate definite integrals if we know antiderivatives)
- Substitution rule

Example: Evaluate

$$\int_0^1 \sqrt{1 + 7x}dx$$

$$\int \frac{x + 3}{\sqrt{x^2 + 6x}}dx$$

$$\int \cos(u).\cos(\sin(u))du$$

Always remember to go back to the original variable.

Integration by parts

Example: u(x)=sin(x), v(x)=x

- Suppose that u(x) and v(x) are functions of x.
- We know the product rule (u(x).v(x))'=u'(x).v(x)+u(x).v'(x)
- ∫(u(x).v(x))' dx= ∫ u'(x).v(x)dx+ ∫ u(x).v'(x)dx
- u(x).v(x)=∫ u'(x).v(x)dx+ ∫ u(x).v'(x)dx (+ c)
- ∫ u.v'dx= u. v - ∫ u'.v dx

Sometimes we drop the +c for a while to make computations neater. Put it back at the end

∫ u.dv = u. v - ∫ v. du

Correct

We use this rule when the function *vdu* is easier to integrate than the original function.

Integration by parts: Evaluate the following:

∫ u.dv = u. v - ∫ v. du

$$\int x.\cos(x)dx$$

$$\int x.\ln(x)dx$$

$$\int \ln(x)du$$

$$\int x^2.\ln(x)dx$$

$$\int e^x.\cos(x)dx$$

We use this rule when the function *vdu* is easier to integrate than the original function and in cases line the last one here (applying twice, use signs).

$\int$ u.dv = u. v - $\int$ v. du

Integration by parts: Tricks

- Write f(x)dx = u.dv (= u(x).(dv/dx). dx)
- The key point is decide what is u and what is dv. Not all possibilities work. (You can always apply the formula, but certain choices will lead to a more complicated problem).
- dv has to be something that you know how to integrate.
- u is everything else.
- $\int$ v. du must be an integral you can eventually solve.
- Sometimes
  - you need to apply parts a few times, (e.g., $f(x)=x^3.e^x$) to make the power of x "dissapear"
  - you need to take u=ln(x), so the ln dissapears. (e.g.,f(x)=ln(x))
  - you need to apply parts twice and use the signs in your favor.(f(x) = $\cos(x).e^x$)

Integration: Tricks

- Always check if your problems gets more and more complicated
- Go slow, write every step, watch signs and constants.
- And remember to go back to the original variable.
- It very important to know the techniques, and it is equally important to know which technique apply. Think before starting to compute.

Example: Evaluate

$$\int \arctan(x)dx$$
$$\int_0^{\ln(3)} \frac{e^x}{e^x-4}\,dx$$
$$\int 2x^3 e^{x^2}\,dx$$
$$\int_{-1}^{0} 15x\sqrt{x+1}\,dx$$
$$\int \sin(\sqrt{x})\,dx$$