MAT 132 Series

8.4 Other convergency tests

The alternating series test

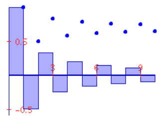
If we have a sequence { a_n },

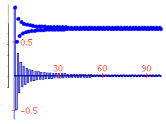
where $a_n > 0$, $a_n \geq a_{n+1}$, and $a_n \to 0$ when $n \to \infty$

then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges





EXAMPLE: Determine if the series below converge

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+2)}{n(n+1)}$$

Recall:
to check a sequence is decreasing,
1. compare terms
2. if the sequence is defined by a function, check derivative is negative.

Determine whether the series below are absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{3} n\pi\right)}{n^2+2} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n (n+2)}{n(n+1)}$$

Definition A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

1 Theorem If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

Determine if the series below convergent, absolutely convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+2)}{n(n+1)} \qquad \sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n$$

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

Series Strategies.

- If $\{a_n\}$does not converge to 0 when $n \to \infty$, then $\sum a_n$ diverges.
- Recall convergence of p-series and geometric series.
- If the series has a form that is similar to a p-series or a geometric series, then try the comparison tests. (Recall that the comparison tests apply only to series with positive terms.)
- Series that involve factorials or other products (including a constant raised to the n-th power) are often conveniently tested using the ratio test. (Note the ratio test does not work with the p-series)
- If the series is alternating, $a_n > 0$, $a_n \leq a_{n+1}$, and $a_n \to 0$ wher $\sum_{n=1}^{\infty} (-1)^n a_n$ $n \to \infty$)
- If $a_n = f(n)$, $f(x) > 0$ for all $x \geq 1$, f is continuous, and decreasing and the integral $\int f(x)\,dx$ is easily evaluated, then try the integral test.

Example

- Approximate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the first 4 terms.
- Estimate the error of the approximation.
- How many terms are required to ensure that the sum is accurate to within 0.0001?
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> **3 Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then
>
> $$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

MAT 132

8.4 Power Series

A power series is in this form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

or

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

The coefficients c_0, c_1, c_2… are constants.

The center "a" is also a constant.

(The first series would be centered at the origin if you graphed it. The second series would be shifted left or right. "a" is the new center.)

EXAMPLES

$$\sum_{n=0}^{\infty} 3\,x^n \qquad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n\,3^n}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sum_{n=0}^{\infty} n!\,x^n$$

A power series in x is a series that can be expressed as

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \cdots$$

The coefficients c_0, c_1, c_2… are constants.

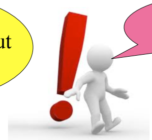
We also use the notation below

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \cdots$$

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \cdots$$

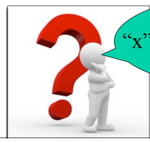
- When we studied infinite series (of constant terms), the key question was..

what is the key question about infinite series?



Does it converge?

- If we replace "x" by a number, a powers series becomes an infinite series. Thus, in the case of power series we ask:



For what values of "x" does the power series converge?

EXAMPLE: Find the values of x for which each of the power series is convergent

$$\sum_{n=0}^{\infty} 3\,x^n \qquad \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, x^n}{n\, 3^n}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sum_{n=0}^{\infty} n!\, x^n$$

A power series in (x-a) is a series that can be expressed as

$$\sum_{n=0}^{\infty} c_n (x-a)x^n = c_0 + c_1(x-a)x + c_2(x-a)x^2 + \cdots + c_n(x-a)^n \cdots$$

The coefficients c_0, c_1, c_2… are constants.

"a" is also a constant

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3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Which of the possibilities of the theorem above hold for each of the power series?

$$\sum_{n=0}^{\infty} 3\,x^n \qquad \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, x^n}{n\, 3^n}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sum_{n=0}^{\infty} n!\, x^n$$

EXAMPLE: Find the values of x for which each of the power series is convergent

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, (x+2)^n}{n\, 3^n}$$

$$\sum_{n=0}^{\infty} 3\,(x-1)^n$$

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

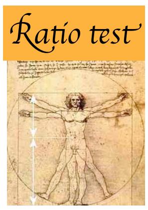
(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

- The number R in (iii) is called the <u>radius of convergence</u> of the power series.
- By convention, R=0 in (i) and R=∞ in (ii)
- The <u>interval of convergen</u>ce is the set of all values of x for which the power series converges.
- There are four possibilities for a power series centered at a.
  - (a-R,a+R)
  - [a-R,a+R)
  - (a-R,a+R]
  - [a-R,a+R]

EXAMPLE: Find the radius of convergence and the interval of convergence



$$\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, (x+2)^n}{n\, 3^n}$$

$$\sum_{n=0}^{\infty} 3\,(x-1)^n$$