MAT132

Directions fields and Euler's method

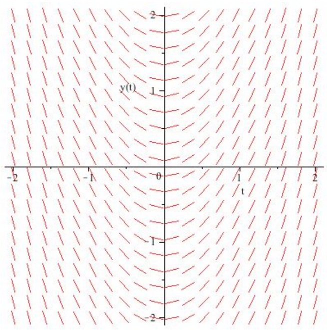
Problem

- Consider the differential equation y'=2x
- Suppose that y=y(x) is a solution.
- What can be say about the slope of the tangent to curve {(x,y(x))} for different values of x?

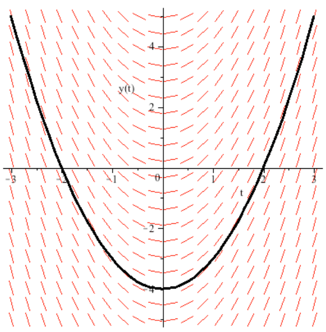
Problem

- Consider the differential equation y'=2x
- For each point (x_0, y_0) of the plane, draw a short piece of a line passing through (x_0, y_0) with slope $2x_0$

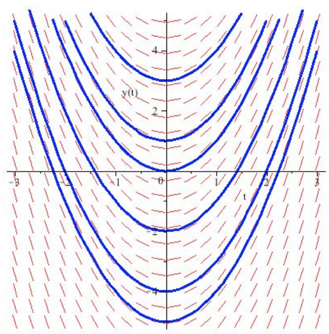
Direction field for y'=2x



Solution of y'=2x, y(0)=-4



Some solutions of y'=2x

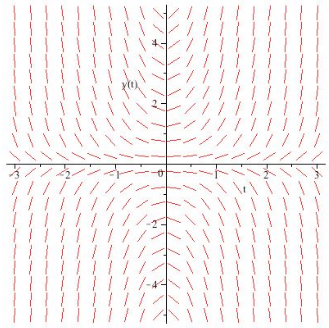


- Does this equation have equilibrium solutions?

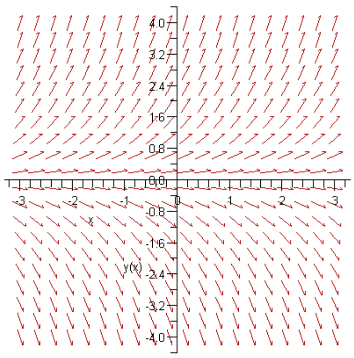
Problem

- Consider the differential equation y'=2.x.y
- For each point (x_0, y_0) of the plane, draw a short piece of a line passing through (x_0, y_0) with slope $2 x_0 y_0$

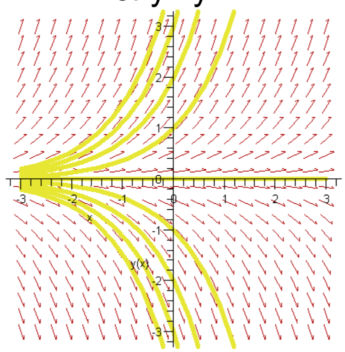
Direction field for y'=2xy



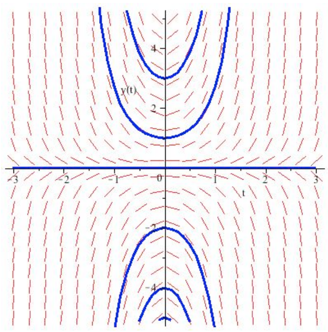
Direction field of y'=y



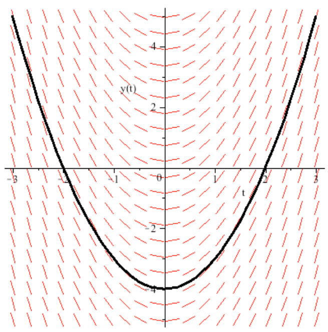
Direction field and some solutions of y'=y



Direction field and some solutions of y'=2xy

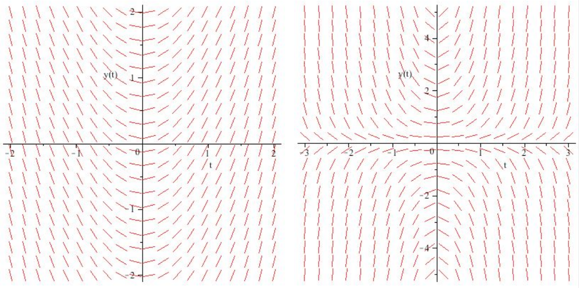


Solution of y'=2x, y(0)=-4

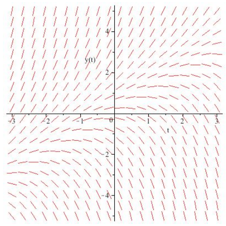


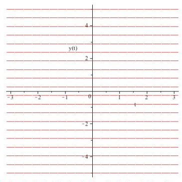
In this case, the solution of the differential equation is one of the antiderivatives of f(x)=2x

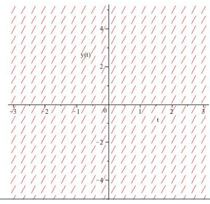
Direction fields for y'=2x and y'=2xy

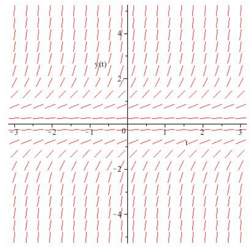


Direction fields. What are the differential equations?

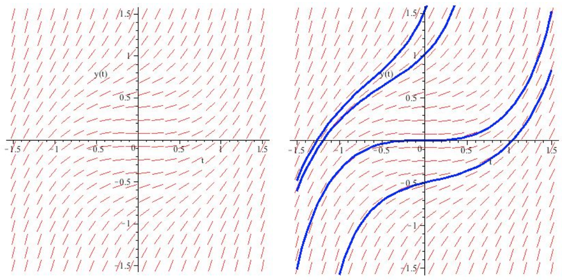








A direction field and some solutions.



Euler's method

Leonhard Euler 1707 - 1783

Euler method

Goal: Find an approximate numerical solution of $y'=2x.y$, $y(1)=2$ at a point close to 1.

First step

Recall: The tangent line is a good approximation to a curve on a small interval.

The equation of the line through (x_0,y_0) with slope m is $y= y_0 +m(x-x_0)$

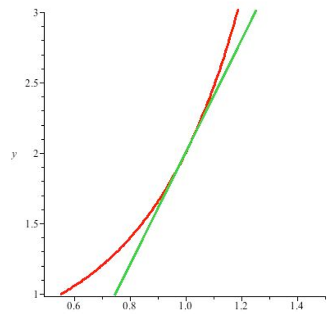
- We begin by approximating solution $y = y(x)$ when x is close to 1.
- The solution (x, y(x)) passes through the point (1, 2).
- The value of y' at (1,2) is $2.\ x_0\ .\ y_0 =2.1.2=4$.
- The tangent line to the solution at the initial point (1,2) is $y = 2+4(x-1)$
- Thus if x_1 is close enough to 0, we can approximate $y(x_1)$ by the line $y_0+m(x_1-x_0)=2+4(x_1-1)$

Euler method

Recall: The tangent line is a good approximation to a curve on a small interval.

Goal: Find an approximate solution of $y'=2x.y$, $y(1)=2$.

- Thus if x_1 is close enough to 0, we can approximate $y(x_1)$ by the line $y_0+m(x_1-x_0)=2+4(x_1-1)$



Euler's method
Given $y'=F(x,y)$, $y(x_0)=y_{0,}$ h

$y_0 = y(x_0)$ initial condition

h stepsize

$x_i = x_0 + i.h$

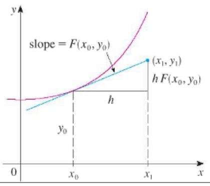
$y_1 = y_0 + hF(x_0, y_0)$

$y_2 = y_1 + hF(x_1, y_1)$

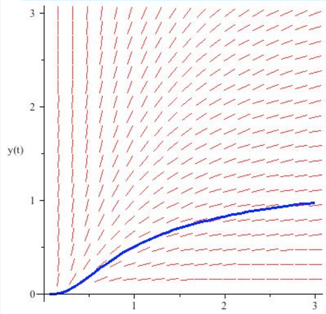
....

$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$

Given (x_i,y_i), and step size h, compute the "next" point (x_{i+1},y_{i+1}),

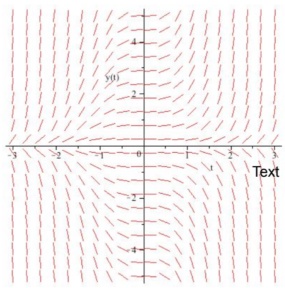


Euler's method
Given $y'=F(x,y)$, $y(x_0)=y_0$, h and n.
Estimate $y(x_n)$



Example: Use Euler's method to estimate y(2) where y(x) is the solution of the initial value problem $y'=y/x^2$, $y(1)=0.5$
1. with step size 1
2. with step size 0.5.
3. with step size 0.2

What is the step size given the best estimation?



Euler's method: Consider the equation $y'=xy^2$, $y(0)=1$.

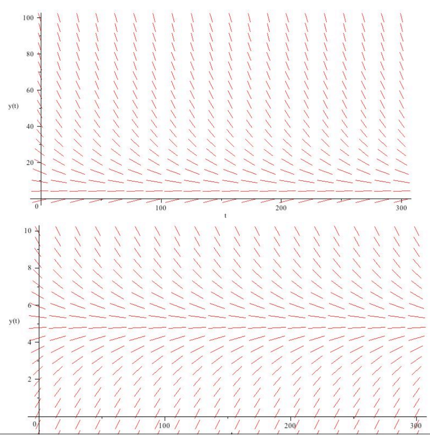
Use Euler's method with step size 0.2 to estimate y(0.6)

1, 1.04, 1.127

Newton's Law of Cooling: the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature.

- A pot of soup had just boiled at 100 degrees C and has to be be served when its temperature is lower than 40 degrees C. The pot is put in a sink full of cold water, (kept running, so that its temperature was roughly constant at 5 degrees C). It is known that the pot cools at a rate of 2 degrees C per minute, when the temperature is 85 degrees. Use Euler's method with step h = 5 to decide whether the soup can be served after 20 minutes.

What can you say about the limiting value of the temperature?



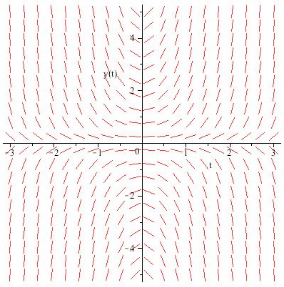
direction fields for the pot of soup problem (different windows)

$$\frac{dy}{dx} = f(x)/g(y)$$

MAT132

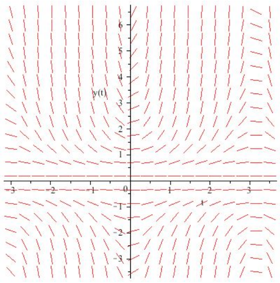
Separable equations

EXAMPLE: Find the solutions of y'=2.x.y



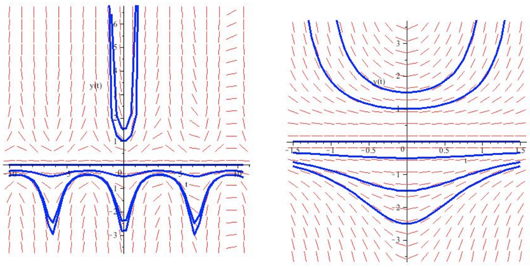
Which are the equilibrium solutions?

EXAMPLE: Find the solutions of y'=sin(x).y^2



What are the equilibrium solutions of y'=sin(x).y^2

Two different "windows" and some solutions of y'=sin(x).y^2



Consider the equation y'=sin(x).y^2

- write f(x)=sin(x), g(y)= y^2
- Then dy/dx = f(x).g(y)
- ∫(1/g(y))dy = ∫f(x)dx

A separable differential equation is a differential equation that can be written as y'= f(x).g(y)

Examples of separable differential equations

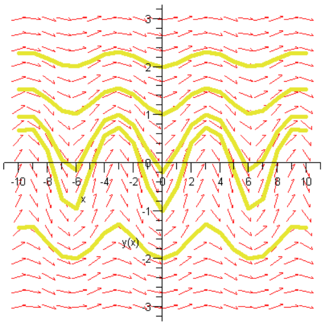
y'=(x+e^x)cos(y)
y'=sin(x).y^2

Examples of non separable differential equations

y'=(x+e^y)cos(x.y)
y'=sin(x+y).y^2

EXAMPLE: Find the solutions of y'=sin(x)/(1+sin(y) + y^2)

Direction field of '=sin(x)/(1+sin(y) + y^2)



Remark

- The solutions of the equation y’=sin(x)/(1+sin(y) + y^2) are all the functions y=y(x) which satisfy the equation y-cos(y)+y^3/3=-cos(x)+C but we it is not possible to give an explicit formula for y.

Example: Solve the differential equations

2 $\dfrac{dy}{dx} = \dfrac{\sqrt{x}}{e^y}$

3. $(x^2+1)y' = xy$

6. $\dfrac{dy}{dt} = \dfrac{e^y \sin^2(t)}{y \sec(t)}$

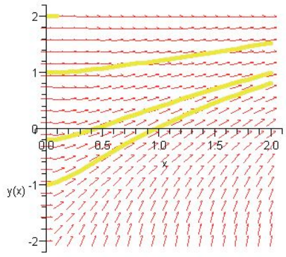
Example: Solve the differential equations

$y(x) = \ln\left(\frac{2}{3}x^{3/2} + \_C1\right)$ 2 $\dfrac{dy}{dx} = \dfrac{\sqrt{x}}{e^y}$

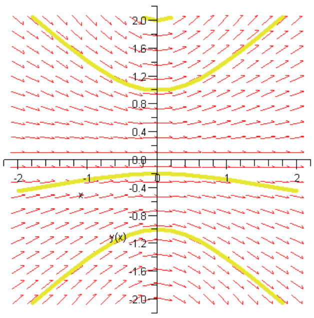
$y(x) = \_C1\sqrt{x^2+1}$ 3. $(x^2+1)y' = xy$

$-e^{-y} y = \frac{1}{3}\sin(t)^3 + c$ 6. $\dfrac{dy}{dt} = \dfrac{e^y \sin^2(t)}{y \sec(t)}$

2. $\dfrac{dy}{dx} = \dfrac{\sqrt{x}}{e^y}$



$(x^2+1)y' = xy$



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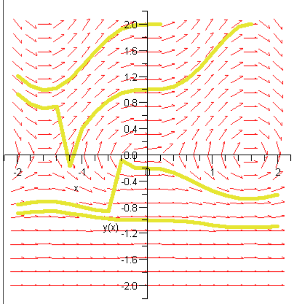
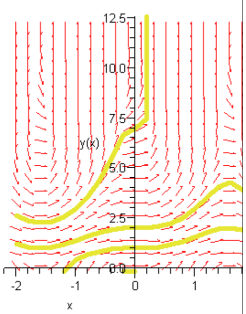
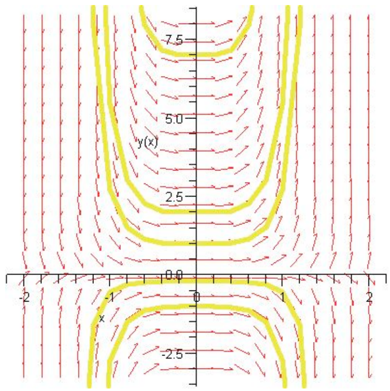


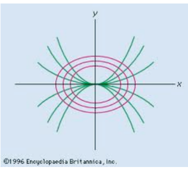


EXAMPLE: Find an equation of the curve that satisfies $dy/dx=4x^3y$ and whose y intercept is 7.

$y = 7\,e^{x^3}$



Orthogonal trajectories



- We are given a family of curves.
- We wish to find curve (or curves) which intersect orthogonally with any member of (whenever they intersect).
- That is, **if** the solution curve (or curves) intersect any member of the given family, the angle between their tangents, at every point of intersection, is π/2.

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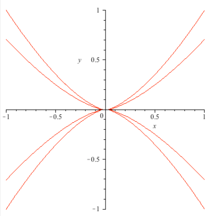
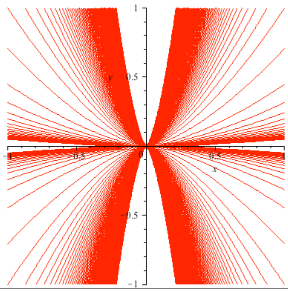
Compute the orthogonal trajectories of the family of curves given by
$y^2=cx^3$
where c is an arbitrary constant.





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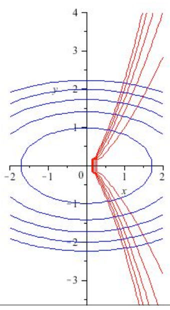
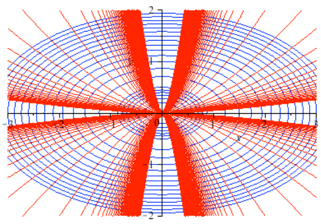
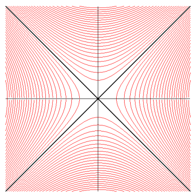


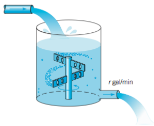


Consider the solutions of the differential equation $dy/dx = x/y$



Mixing problems

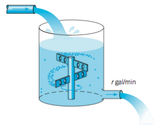
- Goal: build a model that predicts the amount of a substance (salt, paint, etc) in a container with liquid.
- Liquid entering and leaving the container.
- The liquid entering the tank may or may not contain more of the substance dissolved in it.
- Liquid leaving the tank will contain the substance dissolved in it.
- We assume that the concentration of the substance in the liquid is uniform throughout the tank
- Denote by Q(t) the amount of the substance dissolved in the liquid in the tank at any time t.
- **We need to find differential equation that, whose solution is Q(t).**



- Note: we can think of air as a liquid

Mixing problems

- Denote by Q(t) the amount of the substance dissolved in the liquid in the tank at any time t.
- **We need to find differential equation that, whose solution is Q(t).**
- The substance flows into the container at some given rate (input rate), is mixed with the ingredients in the container, and then follows out of the container at some given rate (output rate).
- The **rate of change** of Q, dQ/dt, is equal to the rate at which the substance flows in minus the rate at which the substance flows out.



dQ/dt = (rate in) - (rate out)

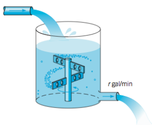
Mixing problems

Rate of change of Q(t) = dQ/dt

Rate at which Q(t) enters the tank =
(flow rate of liquid entering) x (concentration of substance in liquid entering)

Rate at which Q(t) exits the tank =
(flow rate of liquid exiting) x (concentration of substance in liquid exiting)

dQ/dt = (rate in) - (rate out)



Example

- At time t = 0 a tank contains 20lb of salt dissolved in 100 gal of water. Assume that water containing 1/4 lb of salt/gal is entering the tank at a rate of 3 gal/min, and that the well-stirred mixture is draining from the tank at the same rate.
- Set up the initial value problem that describes this flow process.
- Find the amount of salt Q(t) in the tank at any time, and also find the limiting amount of salt that is present after a very long time.