MAT 132

5.7- Integration tricks : Partial Fractions

Techniques of integration

- Properties of integrals
  - Example: ∫(f(x)+g(x))dx=∫f(x)dx+∫g(x)dx
- Tables of integrals
- Rewrite
  - Example: Partial fractions (today's topic)
  - Example: trigonometric identities
- Substitution
  - Example: Trigonometric substitution.

We want to find the antiderivative of a quotient of polynomials P(x)/Q(x) , where the degree of P(x) is smaller than the degree of Q(x)

$$\int \frac{x^2+2x+10}{x(x^2+1)}dx \qquad \int \frac{x^2+2x+10}{x(x^2-1)}dx$$

$$\int \frac{dt}{t^2+4t+4}dt \qquad \int \frac{x+4}{x(x^2+4)}dx$$

$$\int \frac{dt}{t^2+4} \qquad \int \frac{t^2dt}{t^2+4} \qquad \int \frac{x+2}{x^2-2x+2}dx$$

One of the quotients does not satisfy the above requirement, which one?

Always: The degree of the denominator is larger than the degree of the numerator.

We want to find the antiderivative of a quotient of polynomials P(x)/Q(x) , where the degree of P(x) is smaller than the degree of Q(x)

- CASE I: Q(x) is a product of non-repeating linear factors. Example: Q(x) $=x^3-x^2-2x=x(x-2)(x+1)$; x, (x-2) and (x+1) are linear factors
- CASE II: Q(x) is a product of linear factors, some of which can repeat. Example: $Q(x)=x^3-x^2-x+1=(x+1)(x-1)^2$
- CASE III: Q(x) is a product of non-repeating quadratic irreducible factors factors and possibly some linear factos. Example: $Q(x)=(x^2+1)(x^2+4)$
- CASE IV: Q(x) is a product of quadratic irreducible factors (some of which can repeat) Example $Q(x)=(x^2+1)^2$

Note: These are not all possible cases. We will cover Case I,II and III.

The remaining cases can be treated with a combination of the above techniques

EXAMPLE Case I: Evaluate

$$\int \frac{dt}{t^2-1} \qquad \int \frac{x-1}{x^3-x^2-2x}dx$$

1 is a polynomial of degree 0.
t^2-1 is a polynomial of degree 2.
$t^2-1 =(t-1)(t+2)$

x-1 is a polynomial of degree 1.
x^3-x^2-2x is a polynomial of degree 3.
$x^3-x^2-2x =x(x-2)(x+1)$

Always: The degree of the denominator is larger than the degree of the numerator.

- CASE I: Q(x) is a product of non-repeating linear factors. Example: Q(x) $=x^3-x^2-2x=x(x-2)(x+1)$; x, (x-2) and (x+1) are linear factors

EXAMPLE: Evaluate

$$\int \frac{dt}{t^2-1} \qquad \int \frac{x-1}{x^3-x^2-2x}dx$$

$$-\frac{1}{2}\ln(t+1)+\frac{1}{2}\ln(t-1) \qquad \frac{1}{6}\ln(x-2)+\frac{1}{2}\ln(x)-\frac{2}{3}\ln(x+1)$$

$$-\frac{1}{2(t+1)}+\frac{1}{2(t-1)} \qquad \frac{1}{6(x-2)}-\frac{2}{3(x+1)}+\frac{1}{2x}$$

Example Case II: Find

- CASE II: Q(x) is a product of linear factors, some of which can repeat. Example: $Q(x)=x^3-x^2-x+1=(x+1)(x-1)^2$

$$\int \frac{3x+5}{x^3-x^2-x+1}dx$$

$3x+5$ is a polynomial of degree 1.
x^3-x^2-x+1 is a polynomial of degree 3.
$x^3-x^2-x+1 =(x+1)(x-1)^2$

Always: The degree of the denominator is larger than the degree of the numerator.

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Always: The degree of the denominator is larger than the degree of the numerator.

COMPARE TO CASE I
$x-1$ is a polynomial of degree 1.
x^3-x^2-2x is a polynomial of degree 3.
$x^3-x^2-2x =x(x-2)(x+1)$

Example: Find

$$\int \frac{3x+5}{x^3-x^2-x+1}dx$$

$$-\frac{4}{x-1}+\frac{1}{2}\ln(x+1)-\frac{1}{2}\ln(x-1)$$

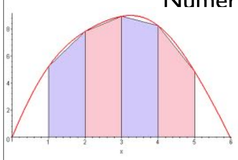
$3x+5$ is a polynomial of degree 1.
x^3-x^2-x+1 is a polynomial of degree 3.
$x^3-x^2-x+1 =(x+1)(x-1)^2$

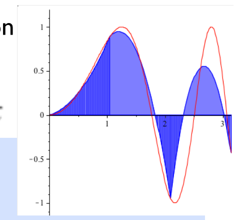
$$-\frac{1}{2(x-1)}+\frac{4}{(x-1)^2}+\frac{1}{2(x+1)}$$

Numerical integration

Estimate

$$\int_a^b f(x)\,dx$$





- Error Trapezoidal rule $-f''(u)(b-a)^3/(12n^2)$

  (where n is the number of intervals, u is some point in (a,b))

- Error Simpson's rule: $-f''''(u)(b-a)^5/(180n^4)$

(where n is the number of intervals, u is some point in (a,b))

Example Case III

- CASE III: Q(x) is a product of non-repeating quadratic irreducible factors factors and possibly some linear factors. Example: $Q(x)=(x^2+1)(x^2+4)$

Always: The degree of the denominator is larger than the degree of the numerator.

$$\int \frac{dt}{t^2+4} \qquad \int \frac{x+4}{x(x^2+4)}dx$$

COMPARE TO CASE II
$3x+5$ is a polynomial of degree 1.
x^3-x^2-x+1 is a polynomial of degree 3.
$x^3-x^2-x+1 =(x+1)(x-1)^2$

COMPARE TO CASE I
$x-1$ is a polynomial of degree 1.
x^3-x^2-2x is a polynomial of degree 3.
$x^3-x^2-2x =x(x-2)(x+1)$

Example Case III

- CASE III: Q(x) is a product of non-repeating quadratic irreducible factors factors and possibly some linear factors. Example: $Q(x)=(x^2+1)(x^2+4)$

Always: The degree of the denominator is larger than the degree of the numerator.

$$\int \frac{dt}{t^2+4} \qquad \int \frac{x+4}{x(x^2+4)}dx$$

Is the integral on the right in case III?

$$\int \frac{t^2dt}{t^2+4}$$

Example Case III

$$\frac{-x+1}{x^2+4} + \frac{1}{x}$$

$$\ln(x) - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{1}{2}x\right)$$

$$\int \frac{x+4}{x(x^2+4)}dx$$

$$\frac{1}{2}\arctan\left(\frac{1}{2}t\right)$$

$$\int \frac{dt}{t^2+4}$$

$$t - 2\arctan\left(\frac{1}{2}t\right)$$

$$\int \frac{t^2 dt}{t^2+4}$$

To "push" the quotient into a form we can apply partial fractions

- completing the square $\int \frac{x+2}{x^2-2x+2}dx$
- long division of polynomials $\int \frac{x^4+2x^2}{x^2-2x+1}\,dx$

$$\int \frac{x^4+2x^2}{x^2-2x+1}\,dx \qquad \frac{1}{3}x^3 + x^2 + 5x + 8\ln(x-1) - \frac{3}{x-1}$$

$$x^2 + 2x + 5 + \frac{8}{x-1} + \frac{3}{(x-1)^2}$$

$$\int \frac{1+x}{x^3-2x^2+5x}\,dx \qquad \frac{1}{5}\ln(x) - \frac{1}{10}\ln(x^2-2x+5) + \frac{3}{5}\arctan\left(\frac{1}{2}x - \frac{1}{2}\right)$$

$$\frac{1}{5x} + \frac{1}{5}\,\frac{7-x}{x^2-2x+5}$$

Techniques of integration

- Properties of integrals
  - Example: ∫(f(x)+g(x))dx=∫f(x)dx+∫g(x)dx **(trial an error)**
- Tables of integrals **(always works)**
- Rewrite
  - Example: Partial fractions **(always works)**
  - Example: trigonometric identities **(trial an error)**
- Substitution **(trial an error)**
  - Example: Trigonometric substitution.

EXAMPLE: Evaluate

$$\int \frac{x^2+2x+10}{x(x^2+1)}dx \qquad \int \frac{x^2+2x+10}{x(x^2-1)}dx$$

$$\int \frac{1}{e^x - 3e^{3x}}dx$$

$$\int \frac{dt}{t^2+4t+4}dt \qquad \int \frac{x+2}{x^2-2x+2}dx$$

$$10\ln(x) - \frac{9}{2}\ln(x^2+1) + 2\arctan(x) \qquad -10\ln(x) + \frac{9}{2}\ln(x+1) + \frac{13}{2}\ln(x-1)$$

$$\frac{-9x+2}{x^2+1} + \frac{10}{x} \qquad \frac{13}{2(x-1)} + \frac{9}{2(x+1)} - \frac{10}{x}$$

$$\int \frac{x^2+2x+10}{x(x^2+1)}dx \qquad \int \frac{x^2+2x+10}{x(x^2-1)}dx$$

$$\int \frac{1}{e^x - 3e^{3x}}dx \qquad -\frac{1}{e^x} + \frac{1}{2}\ln(e^x+1) - \frac{1}{2}\ln(e^x-1)$$

$$\int \frac{dt}{t^2+4t+4}dt \qquad -\frac{1}{t+2}$$

$$\frac{1}{2}\ln(x^2-2x+2) + 3\arctan(x-1)$$

$$\int \frac{dx}{x^2-2x+2} \qquad \int \frac{x+2}{x^2-2x+2}dx$$

$$\arctan(x-1)$$