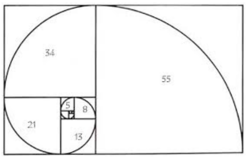
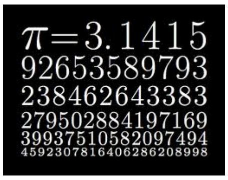
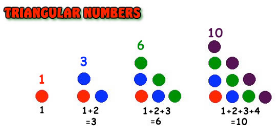
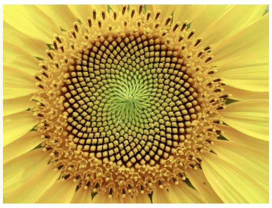




MAT 132 8.1 Sequences



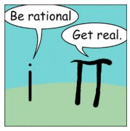


A sequence is a an infinite list of numbers written in order.

$$\{a_n\} = \{a_1,\ a_2,\ a_3,\ \ldots\ ,\ a_n,\ \ldots\}$$

nth term

Examples
{1,1,1,1,..}
{1,2,3,..}
{½,-2/3,3/4,-4/5...}
{√2,√3,√4,... }
{1,4,1,5,9,2…}



Find the n-th term of the above sequences.

An (infinite) sequence is a an infinite list of numbers written in order.
An (infinite) sequence is thus a function, where the domain is the set of positive integers and the range is the real numbers.

$$\{a_n\} = \{a_1,\ a_2,\ a_3,\ \ldots\ ,\ a_n,\ \ldots\}$$

nth term

Examples
{1,1,1,1,..}
{1,2,3,..}
{½,-2/3,3/4,-4/5...}
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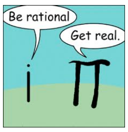
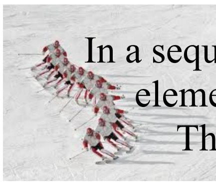


Find the n-th term of the above sequences.



In a sequence order matters and elements can be repeated. Thus sequence ≠ set

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A sequence is defined explicitly if there is a formula that allows you to find individual terms independently

Example: Consider the sequence given by

$a_n = 3^n$.

The first, second, third and fourth terms of this sequences are

$a_1 = 3^1 = 3,$

$a_2 = 3^2 = 9,$

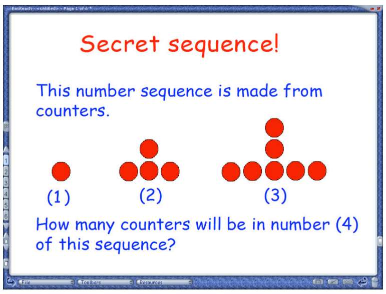
$a_3 = 3^3 = 27,$

$a_4 = 3^4 = 81$

Example: $a_n = \dfrac{(-1)^n}{n^2+1}$

To find the 100th term, plug 100 in for n: $a_{100} = \dfrac{(-1)^{100}}{100^2+1} = \dfrac{1}{10001}$

Challenge: Find the 100-th term of the sequence below.



source: http://mrjosephsprecalculusblog.blogspot.com/

A sequence is defined recursively if there is a formula that relates a_n to previous terms.

Example 1: $b_1 = 4 \qquad b_n = b_{n-1} + 2 \quad \text{for all } n \geq 2$
Example 2: Fibonacci sequence $b_1=1, b_2=1, b_n=b_{n-1}+b_{n-2}$ for $n \geq 3$
Example 3: Collatz sequences

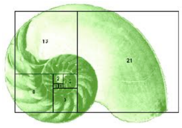
Example 1: $b_1 = 4$

$b_2 = b_1 + 2 = 6$

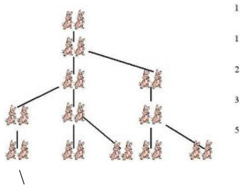
$b_3 = b_2 + 2 = 8$

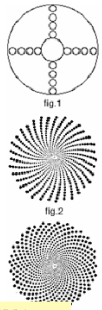
$b_4 = b_3 + 2 = 10$

Can you give an explicit definition of this sequence?

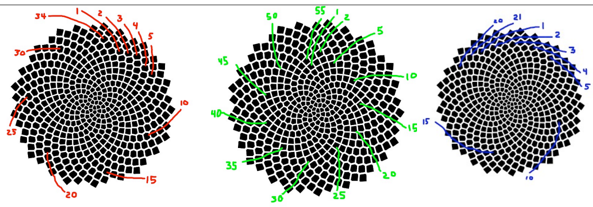


Fibonacci sequence in nature

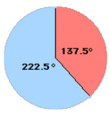




The golden angle, is 137.5 degrees.



1,1,2,3,5,8,13,21,34,55,89...

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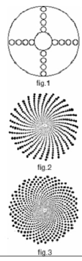


Example of sequences defined recursively: Collatz sequences

f(n) = n/2 if n is even
3n+1 otherwise

Conjecture: No matter which number you start from, the sequence always reaches 1

Start with a positive integer, say, 10,
$a_1=2$
$a_2=f(a_1)=5$
$a_3=f(a_2)=16$
and so on.
This is a recursively defined sequence.
(Starting at a different number, you'll obtained a different sequence)

2009: The Collatz algorithm has been tested and found to always reach 1 for all numbers up to 5.4×10^{18}

An arithmetic sequence is a sequence such that the difference between consecutive terms is constant.

Examples: $-5, -2, 1, 4, 7, \ldots \qquad d = 3$

$\ln 2, \ln 6, \ln 18, \ln 54, \ldots \qquad d = \ln 6 - \ln 2 = \ln \frac{6}{2} = \ln 3$

Arithmetic sequences can be defined recursively: $a_n = a_{n-1} + d$

or explicitly: $a_n = a_1 + d(n-1)$

A sequence is geometric if the quotient of consecutive terms is constant. That is consecutive terms have the same ratio.

Example: $1, -2, 4, -8, 16, \ldots \qquad r = -2$

$10^{-2}, 10^{-1}, 1, 10, \ldots \qquad r = \frac{10^{-1}}{10^{-2}} = 10$

Geometric sequences can be defined recursively: $a_n = a_{n-1} \cdot r$

or explicitly: $a_n = a_1 \cdot r^{n-1}$

A sequence is defined explicitly if there is a formula that allows you to find individual terms independently.

Ex: $a_n=n/(n^2+1)$

Any real-valued function defined on the positive real yields a sequence (explicitly defined).

Example: $f(x)=(x+2)^{½}$
n-th of the sequence: $a_n=(n+2)^{½}$

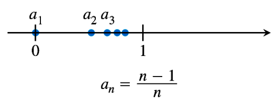
A sequence is defined recursively if there is a formula that relates a_n to previous terms.

An arithmetic sequence has a common difference between terms.

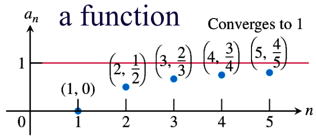
An geometric sequence has a common ratio between terms.

Write the first terms of the sequence $a_n = \frac{n-1}{n}$

Plot these terms on a number line

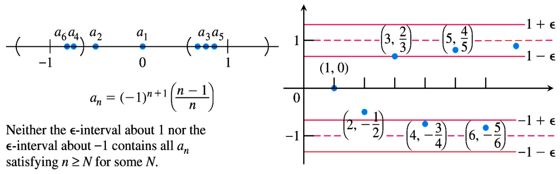


Plot the sequence as a function



The terms in this sequence get closer and closer to 1.
The sequence CONVERGES to 1.

Consider the sequence $a_n = \frac{(-1)^{n+1}(n-1)}{n}$



The terms in this sequence do not get close to any (single) number when n grows.

The sequence $\{a_n\}$ *converges to L* if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.
EXAMPLE

$$a_n = \frac{n-1}{n}$$

Otherwise, that is if $\{a_n\}$ does not converges to any number, we say that $\{a_n\}$ *diverges*.
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$$a_n = \frac{(-1)^{n+1}(n-1)}{n}$$

The sequence $\{a_n\}$ *converges to L* if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.
We write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

Otherwise, that is if $\{a_n\}$ does not converges to any number, we say that $\{a_n\}$ *diverges*.
EXAMPLE

A sequence is geometric if the quotient of consecutive terms is constant. That is consecutive terms have the same ratio.

Example: $1, -2, 4, -8, 16, \ldots$ $\quad r = -2$

$10^{-2}, 10^{-1}, 1, 10, \ldots$ $\quad r = \frac{10^{-1}}{10^{-2}} = 10$

Geometric sequences can be defined recursively: $a_n = a_{n-1} \cdot r$

or explicitly: $a_n = a_1 \cdot r^{n-1}$

Can you find examples of convergent geometric sequence? And of diverent geometric sequences?

A sequence is geometric if the quotient of consecutive terms is constant. That is consecutive terms have the same ratio.

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Can you find examples of convergent geometric sequence? And of diverent geometric sequences?

Determine whether the sequences below are convergent.

I. $a_n = 3^n$,
II. $a_n = (½)^n$
III. $a_n = (-1)^n$
IV. $a_n = (-2)^n$
V. $a_n = (-0.1)^n$

7 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

2 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

- Examples: Study whether the sequences below converge using the theorem above (if possible)

$$a_n = \frac{n-1}{n}$$

$$a_n = \frac{(-1)^{n+1}(n-1)}{n}$$

- Example: The above theorem cannot be used to prove that the sequence $a_n = 1/n!$ converges. Why?

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Example: Below is the n-th term of some sequences
Determine whether the corresponding sequences converge and if so, find the limit.

I. $a_n = 1/n$
II. $a_n = 1/n + 3(n+1)/n^2$
III. $b_n = (a_n)^2$ (a_n defined in the line above).
IV. $a_n = n!/(n+1)!$
V. $a_n = (n+1)!/n!$
VI. $a_n = 1/\ln(n)$.
VII. $a_n = n/\ln(n)$.

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

- Example: Use the “squeeze theorem” above to determine whether the sequence $\{a_n\}$ defined by $a_n = (n^2+1)/n^3$ is converges and if so, find the limit.

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Give examples of
- Increasing, convergent sequences.
- Decreasing convergent sequences.
- Increasing divergent sequences.
- Decreasing divergent sequences.
- Convergent sequences that are not increasing and not decreasing
- Divergent sequences that are not increasing and not decreasing

Types of sequences

Defined explicitly Ex: $a_n=n/(n^2+1)$
Defined recursively Ex: $a_1=1$, $a_2=1$, $a_n=a_{n-1}+a_{n-2}$, $n\geq3$

Defined by function Example: $f(x)=(x+2)^{½}$ $a_n=(n+2)^{½}$

Convergent $a_n=1/n$
Divergent, $a_n=n$ or $a_n=(-1)^n$

Arithmetic $a_n=a_1+ (n-1) . d$
Geometric $a_n=a_1. r^{n-1}$

Increasing $a_n=(1-1/n)$
Decreasing $a_n=1/n$