Math 132 Final Exam Practice Problems
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(1) Evaluate the integral $\int (\ln(x))^3\,dx$.

Let $u = (\ln(x))^3$ and $dv = dx$. Then $du = 3(\ln(x))^2\frac{1}{x}dx$ and $v = x$. Integrate by parts:

$$\begin{aligned}\int (\ln(x))^3\,dx &= uv - \int v\,du\\ &= x(\ln(x))^3 - \int x3(\ln(x))^2\frac{1}{x}\,dx\\ &= x(\ln(x))^3 - 3\int (\ln(x))^2\,dx.\end{aligned}$$

Now let $u = (\ln(x))^2$, and $dv = dx$. Then $du = 2(\ln(x))\frac{1}{x}dx$ and $v = x$. Integrate by parts again:

$$\begin{aligned}\int (\ln(x))^2\,dx &= x(\ln(x))^2 - \int x2(\ln(x))\frac{1}{x}\,dx\\ &= x(\ln(x))^2 - 2\int \ln(x)\,dx.\end{aligned}$$

Finally, let $u = \ln(x)$ and $dv = dx$. Then $du = \frac{1}{x}dx$ and $v = x$. Integrate by parts:

$$\int \ln(x)\,dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - \int dx = x\ln(x) - x + C.$$

Now put it all together:

$$\begin{aligned}\int (\ln(x))^3\,dx &= x(\ln(x))^3 - 3\int (\ln(x))^2\,dx\\ &= x(\ln(x))^3 - 3\left(x(\ln(x))^2 - 2\int \ln(x)\,dx\right)\\ &= x(\ln(x))^3 - 3\big(x(\ln(x))^2 - 2(x\ln(x) - x + C)\big)\\ &= x(\ln(x))^3 - 3x(\ln(x))^2 + 6x\ln(x) - 6x + C.\end{aligned}$$

(2) Evaluate the integral $\int \cos^4(3x)\,dx$.

We use the power reduction formula $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$. This gives

$$\begin{aligned}\int \cos^4(3x)\,dx &= \int \left(\frac{1+\cos(6x)}{2}\right)^2\,dx\\ &= \frac{1}{4}\int 1 + 2\cos(6x) + \cos^2(6x)\,dx\\ &= \frac{1}{4}\int dx + \frac{1}{2}\int \cos(6x)\,dx + \frac{1}{4}\int \cos^2(6x)\,dx\\ &= \frac{1}{4}x + \frac{1}{12}\sin(6x) + \frac{1}{4}\int \cos^2(6x)\,dx.\end{aligned}$$

Applying the power reduction formula again, we get

$$\begin{aligned}
\frac{1}{4}x + \frac{1}{12}\sin(6x) + \frac{1}{4}\int \cos^2(6x)\,dx &= \frac{1}{4}x + \frac{1}{12}\sin(6x) + \frac{1}{4}\int \frac{1+\cos(12x)}{2}\,dx \\
&= \frac{1}{4}x + \frac{1}{12}\sin(6x) + \frac{1}{8}\int 1+\cos(12x)\,dx \\
&= \frac{1}{4}x + \frac{1}{12}\sin(6x) + \frac{1}{8}\int dx + \frac{1}{8}\int \cos(12x)\,dx \\
&= \frac{1}{4}x + \frac{1}{12}\sin(6x) + \frac{1}{8}x + \frac{1}{8\cdot 12}\sin(12x) + C.
\end{aligned}$$

(3) Evaluate the integral $\int_0^\pi e^x \sin(\pi - x)\,dx$.

We solve the indefinite integral first. Let $u = \sin(\pi - x)$ and $dv = e^x dx$. Then $du = -\cos(\pi - x)dx$ and $v = e^x$. Integrate by parts:

$$\int e^x \sin(\pi - x)\,dx = e^x \sin(\pi - x) + \int e^x \cos(\pi - x)\,dx.$$

Now let $u = \cos(\pi - x)$ and $dv = e^x dx$. Then $du = \sin(\pi - x)dx$ and $v = e^x$. Integrate by parts again:

$$\int e^x \cos(\pi - x)\,dx = e^x \cos(\pi - x) - \int e^x \sin(\pi - x)\,dx.$$

Putting these together, we get

$$\int e^x \sin(\pi - x)\,dx = e^x \sin(\pi - x) + e^x \cos(\pi - x) - \int e^x \sin(\pi - x)\,dx.$$

Adding $\int e^x \sin(\pi - x)\,dx$ to both sides and dividing by two gives

$$\int e^x \sin(\pi - x)\,dx = \frac{e^x \sin(\pi - x) + e^x \cos(\pi - x)}{2}.$$

Thus the definite integral is

$$\begin{aligned}
\int_0^\pi e^x \sin(\pi - x)\,dx &= \left.\frac{e^x \sin(\pi - x) + e^x \cos(\pi - x)}{2}\right|_0^\pi \\
&= \frac{e^\pi(0) + e^\pi(1)}{2} - \frac{e^0(0) + e^0(-1)}{2} \\
&= \frac{e^\pi + 1}{2}.
\end{aligned}$$

(4) Evaluate the integral $\int \frac{x}{(2x+5)(x-2)}\,dx$.

We use partial fractions.

$$\frac{x}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}.$$

So, $A(x-2) + B(2x+5) = x$, giving

$$\begin{aligned}
A + 2B &= 1 \\
-2A + 5B &= 0.
\end{aligned}$$

Therefore, $B = \frac{2}{5}A$, and $A + \frac{4}{5}A = 1$. Thus, $A = \frac{5}{9}$ and $B = \frac{2}{9}$. So the integral is

$$\begin{aligned}\int \frac{x}{(2x+5)(x-2)}\,dx &= \int \frac{5/9}{2x+5} + \frac{2/9}{x-2}\,dx \\ &= \frac{5}{9}\int \frac{1}{2x+5}\,dx + \frac{2}{9}\int \frac{1}{x-2}\,dx \\ &= \frac{5}{18}\ln|2x+5| + \frac{2}{9}\ln|x-2| + C.\end{aligned}$$

(5) Evaluate integral $\int \arctan(1/x)\,dx$.

Let $u = \arctan(1/x)$ and $dv = dx$. Then $du = \frac{1}{1+(1/x)^2}\frac{-1}{x^2}dx$ and $v = x$. Integrate by parts:

$$\begin{aligned}\int \arctan(1/x)\,dx &= x\arctan(1/x) - \int x\frac{1}{1+(1/x)^2}\frac{-1}{x^2}\,dx \\ &= x\arctan(1/x) - \int \frac{-x}{x^2(1+(1/x)^2)}\,dx \\ &= x\arctan(1/x) + \int \frac{x}{x^2+1}\,dx.\end{aligned}$$

Now do u-substitution: let $u = x^2+1$. Then $du = 2xdx$. So the integral becomes

$$\begin{aligned}\int \arctan(1/x)\,dx &= x\arctan(1/x) + \frac{1}{2}\int \frac{1}{u}\,du \\ &= x\arctan(1/x) + \frac{1}{2}\ln|u| + C \\ &= x\arctan(1/x) + \frac{1}{2}\ln|x^2+1| + C.\end{aligned}$$

(6) Evaluate the integral $\int xe^{-3x}\,dx$.

Let $u = x$, $dv = e^{-3x}dx$. Then $du = dx$ and $v = -\frac{1}{3}e^{-3x}$. Integrate by parts:

$$\begin{aligned}\int xe^{-3x}\,dx &= -\frac{x}{3}e^{-3x} + \int \frac{1}{3}e^{-3x}\,dx \\ &= -\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x} + C.\end{aligned}$$

(7) Evaluate the integral $\int \cos^5(3x)\,dx$.

Using the Pythagorean theorem, write the integral as $\int (1-\sin^2(3x))^2\cos(3x)\,dx$. Now apply u-substitution: Let $u = \sin(3x)$. Then $du = 3\cos(3x)\,dx$, so the integral becomes

$$\begin{aligned}\int (1-\sin^2(3x))^2\cos(3x)\,dx &= \frac{1}{3}\int (1-u^2)^2\,du \\ &= \frac{1}{3}\int 1 - 2u^2 + u^4\,du \\ &= \frac{1}{3}\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C \\ &= \frac{1}{3}\left(\sin(3x) - \frac{2}{3}\sin^3(3x) + \frac{1}{5}\sin^5(3x)\right) + C.\end{aligned}$$

In general, if you want to compute the integral of an even power of sin or cos, you will apply the power reduction formula like in problem (2). Odd powers of sin and cos are integrated as in this problem.

(8) Prove that the area of a circle with radius r is πr^2.

The circle of radius r about the origin is given by the formula $x^2 + y^2 = r^2$, so the top half of the circle is given by the function

$$y = \sqrt{r^2 - x^2}.$$

The area of the circle of radius r is then given by

$$A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx.$$

We use trigonometric substitution. Let $x = r\sin\theta$. Then $dx = r\cos\theta d\theta$. If $x = -r$, then $\sin\theta = -1$, so $\theta = -\pi/2$. If $x = r$, then $\sin\theta = 1$, so $\theta = \pi/2$. The integral becomes

$$\begin{aligned}
A &= 2\int_{-\pi/2}^{\pi/2} \sqrt{r^2 - (r\sin\theta)^2} r\cos\theta\, d\theta \\
&= 2\int_{-\pi/2}^{\pi/2} r\sqrt{1 - \sin^2\theta} r\cos\theta\, d\theta \\
&= 2\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\, d\theta.
\end{aligned}$$

We then apply the power reduction formula to get

$$\begin{aligned}
A &= 2r^2\int_{-\pi/2}^{\pi/2} \frac{1 + \cos(2\theta)}{2}\, d\theta \\
&= r^2\int_{-\pi/2}^{\pi/2} d\theta + r^2\int_{-\pi/2}^{\pi/2} \cos(2\theta)\, d\theta \\
&= r^2\ \theta\Big|_{-\pi/2}^{\pi/2} + \frac{r^2}{2}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2} \\
&= r^2(\pi/2 - (-\pi/2)) + \frac{r^2}{2}(\sin(\pi) - \sin(-\pi)) \\
&= r^2\pi + \frac{r^2}{2}(0 - 0) \\
&= \pi r^2.
\end{aligned}$$

Alternatively, you can use polar coordinates and the area formula: In polar coordinates, the circle of radius r_0 is given by the function $r(\theta) = r_0$. The area formula then gives

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} r_0^2\, d\theta \\
&= \frac{r_0^2}{2}\int_0^{2\pi} d\theta \\
&= \frac{r_0^2}{2}\ \theta\Big|_0^{2\pi} \\
&= \frac{r_0^2}{2}(2\pi - 0) \\
&= \pi r_0^2.
\end{aligned}$$

(9) Evaluate the integral $\int \frac{\sqrt{16-x^2}}{x^2}\,dx$.

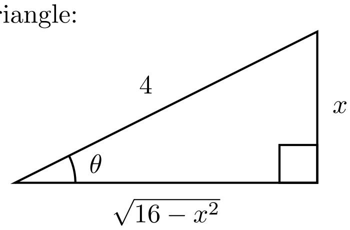
We use trig substitution. Let $x = 4\sin\theta$. Then $dx = 4\cos\theta d\theta$. So the integral becomes

$$\begin{aligned}
\int \frac{\sqrt{16-x^2}}{x^2}\,dx &= \int \frac{\sqrt{16-16\sin^2\theta}}{16\sin^2\theta}\,4\cos\theta\,d\theta \\
&= \int \frac{4\cos\theta}{16\sin^2\theta} 4\cos\theta\,d\theta \\
&= \int \cot^2\theta\,d\theta.
\end{aligned}$$

Dividing both sides of of $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$ gives $1+\cot^2\theta = \csc^2\theta$, so the integral becomes

$$\int \csc^2\theta - 1\,d\theta = -\cot\theta - \theta + C.$$

Now we put the answer back in terms of x. Since $\sin\theta = \frac{x}{4}$, we have the following right triangle:



So, $-\cot\theta - \theta + C = \frac{-\sqrt{16-x^2}}{x} - \arcsin(x/4) + C$.

(10) Consider a real number p. Find the values of p for which the integral $\int_2^\infty x^p\,dx$ converges, and evaluate the integral for those values of p.

Since the integral is improper, we need to write it as a limit:

$$\int_2^\infty x^p\,dx = \lim_{t\to\infty}\int_2^t x^p\,dx.$$

If $p \neq -1$, the integral is

$$\begin{aligned}
\lim_{t\to\infty}\int_2^t x^p\,dx &= \lim_{t\to\infty} \frac{1}{p+1}x^{p+1}\Big|_2^t \\
&= \lim_{t\to\infty}\frac{t^{p+1}-2^{p+1}}{p+1} \\
&= \frac{\lim_{t\to\infty} t^{p+1}}{p+1} - \frac{2^{p+1}}{p+1}.
\end{aligned}$$

If $p > -1$, then $\lim_{t\to\infty} t^{p+1} = \infty$ and the integral is divergent. If $p < -1$, then $\lim_{t\to\infty} t^{p+1} = 0$, and the integral is convergent, and the value is $-\frac{2^{p+1}}{p+1}$.

This leaves the case where $p = -1$. In this case,

$$\begin{aligned}
\lim_{t\to\infty}\int_2^t x^p\,dx &= \lim_{t\to\infty}\int_2^t \frac{1}{x}\,dx \\
&= \lim_{t\to\infty} \ln(x)|_2^t \\
&= \lim_{t\to\infty} \ln(t) - \ln(2) \\
&= \infty.
\end{aligned}$$

So, in summary, the integral converges for all $-\infty < p < -1$, and diverges otherwise. When it converges, the value is $-\frac{2^{p+1}}{p+1}$.

(11) Evaluate the integral or show it is divergent

(a) $\int_2^\infty \frac{2}{(2x+3)^4}\,dx$.
We use the comparison test, comparing the function to $\frac{2}{x^4}$. Since $(2x+3)^4 \geq x^4$, it follows that

$$0 \leq \frac{2}{(2x+3)^4} \leq \frac{2}{x^4}.$$

Since $\int_2^\infty \frac{2}{x^4}\,dx = 2\int_2^\infty \frac{1}{x^4}\,dx$ converges (p-test for $p = -4$, or problem (10),) So too does $\int_2^\infty \frac{2}{(2x+3)^4}\,dx$.

(b) $\int_{-\infty}^0 e^{-3x}\,dx$.
You can use the comparison test, comparing e^{-3x} to the function $y = -x$, or try to compute the integral directly:

$$\begin{aligned}\int_{-\infty}^0 e^{-3x}\,dx &= \lim_{t\to-\infty}\int_t^0 e^{-3x}\,dx\\ &= \lim_{t\to\infty} \left.\frac{-e^{-3x}}{3}\right|_{-t}^0\\ &= \lim_{t\to\infty} \frac{-1}{3} + \frac{e^{3t}}{3}\\ &= \infty.\end{aligned}$$

So, the integral diverges.

(c) $\int_{-10}^1 \frac{x}{\sqrt{x+10}}\,dx$.
Let us compute the integral directly. There's an asymptote at $x = -10$, which is why the integral is improper. We have:

$$\int_{-10}^1 \frac{x}{\sqrt{x+10}}\,dx = \lim_{t\to-10^+}\int_t^1 \frac{x}{\sqrt{x+10}}\,dx.$$

Let $u = \sqrt{x+10}$. Then $du = \frac{dx}{2\sqrt{x+10}}$, and $x = u^2 - 10$. If $x = t$, then $u = \sqrt{t+10}$. If $x = 1$, then $u = \sqrt{11}$. So, the integral is

$$\begin{aligned}\lim_{t\to-10^+}\int_t^1 \frac{x}{\sqrt{x+10}}\,dx &= \lim_{t\to-10^+} 2\int_{\sqrt{t+10}}^{\sqrt{11}} (u^2-10)\,du\\ &= \lim_{t\to-10^+} 2\left(\frac{u^3}{3} - 10u\right)_{\sqrt{t+10}}^{\sqrt{11}}\\ &= \lim_{t\to-10^+} 2\left(\frac{\sqrt{11}^3}{3} - 10\sqrt{11}\right) - 2\left(\frac{\sqrt{t+10}^3}{3} - 10\sqrt{t+10}\right)\\ &= 2\left(\frac{\sqrt{11}^3}{3} - 10\sqrt{11}\right)\\ &= \frac{-8\sqrt{11}}{3}.\end{aligned}$$

(d) $\int_1^e \frac{1}{x\ln\sqrt{x}}\,dx$.

The asymptote occurs at $x = 1$. Let us try to evaluate the integral directly:

$$\begin{aligned}\int_1^e \frac{1}{x \ln \sqrt{x}}\, dx &= \lim_{t\to 1^+} \int_t^e \frac{1}{x \ln \sqrt{x}}\, dx \\ &= \lim_{t\to 1^+} \int_t^e \frac{1}{x \frac{1}{2} \ln x}\, dx \\ &= 2 \lim_{t\to 1^+} \int_t^e \frac{1}{x \ln x}\, dx.\end{aligned}$$

Let $u = \ln x$. Then $du = dx/x$, so the integral becomes

$$\begin{aligned}2 \lim_{t\to 1^+} \int_t^e \frac{1}{x \ln x}\, dx &= 2 \lim_{t\to 1^+} \int_{\ln t}^{\ln e} \frac{1}{u}\, du \\ &= 2 \lim_{t\to 1^+} \ln(u)\Big|_{\ln t}^{\ln e} \\ &= 2 \lim_{t\to 1^+} \ln(\ln e) - \ln(\ln t) \\ &= \infty,\end{aligned}$$

since as $t \to 1$, we have $\ln t \to 0$, and so $\ln(\ln t) \to -\infty$. Thus, the integral is divergent.

(12) Find the length of the polar curve $r = e^{2\theta}$, $0 \le \theta \le 2\pi$.

Since $\frac{dr}{d\theta} = 2e^{2\theta}$, the length of the polar curve is given by the integral

$$\begin{aligned}\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta &= \int_0^{2\pi} \sqrt{\left(e^{2\theta}\right)^2 + \left(2e^{2\theta}\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta \\ &= \sqrt{5} \int_0^{2\pi} e^{2\theta}\, d\theta \\ &= \sqrt{5}\, \frac{e^{2\theta}}{2}\Big|_0^{2\pi} \\ &= \frac{e^{4\pi} - 1}{2}\sqrt{5}.\end{aligned}$$

(13) Find the area of the region that lies inside the curve $r = 2(\sqrt{2} + \sin\theta)$ and outside the curve $r = 3\sqrt{2}$. (note the corrected typos)

The two curves intersect when $3\sqrt{2} = 2(\sqrt{2} + \sin\theta)$, i.e. when $\frac{\sqrt{2}}{2} = \sin\theta$, i.e. when $\theta = \pi/4$ and $\theta = 3\pi/4$. The value of $2(\sqrt{2} + \sin\theta)$ is larger than $3\sqrt{2}$ when $\pi/4 < \theta < 3\pi/4$, so this is the interval on which the curve $r = 2(\sqrt{2} + \sin\theta)$ lies outside the curve $r = 3\sqrt{2}$.

The desired area is therefore

$$\begin{aligned}
A &= \int_{\pi/4}^{3\pi/4} \frac{1}{2}\left(2(\sqrt{2}+\sin\theta)\right)^2 d\theta - \int_{\pi/4}^{3\pi/4} \frac{1}{2}\left(3\sqrt{2}\right)^2 d\theta \\
&= \int_{\pi/4}^{3\pi/4} (4 + 4\sqrt{2}\sin\theta + 2\sin^2\theta)\, d\theta - \int_{\pi/4}^{3\pi/4} 4\, d\theta \\
&= 4\sqrt{2}\int_{\pi/4}^{3\pi/4} \sin\theta\, d\theta + 2\int_{\pi/4}^{3\pi/4} \sin^2\theta\, d\theta \\
&= -4\sqrt{2}\cos\theta\Big|_{\pi/4}^{3\pi/4} + \int_{\pi/4}^{3\pi/4} (1-\cos(2\theta))\, d\theta \\
&= -4\sqrt{2}\,(\cos(3\pi/4) - \cos(\pi/4)) + \left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_{\pi/4}^{3\pi/4} \\
&= -4\sqrt{2}\left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) + \left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_{\pi/4}^{3\pi/4} \\
&= -4\sqrt{2}\left(-\sqrt{2}\right) + \left(3\pi/4 - \frac{\sin(3\pi/2)}{2}\right) - \left(\pi/4 - \frac{\sin(\pi/2)}{2}\right) \\
&= 8 + \left(3\pi/4 + \frac{1}{2}\right) - \left(\pi/4 - \frac{1}{2}\right) \\
&= 9 + \pi/2.
\end{aligned}$$

(14) Find the area of the region bounded by the given curves $y = 1/x$, $y = x^3$, $y = 0$, and $x = 2$.

First, draw the region. The two curves $y = 1/x$ and $y = x^3$ intersect at $x = 1$, so the area of the region is given by

$$\begin{aligned}
\int_0^1 x^3\, dx + \int_1^2 \frac{dx}{x} &= \frac{x^4}{4}\Bigg|_0^1 + \ln(x)\Big|_1^2 \\
&= \frac{1}{4} + \ln(2).
\end{aligned}$$

(15) Find the area of the region bounded by the given curves $x + y = 0$ and $x = y^2 + 3y$.

The first curve is a line through the origin with slope -1, and the second is a sideways parabola with roots $y = 0$ and $y = -3$. To find where the two curves intersect, plug $x = -y$ into the equation for the parabola: $-y = y^2 + 3y$, i.e. $y^2 + 4y = 0$, i.e. $y = 0$ or $y = -4$. Thus, the area of the region is given by

$$\begin{aligned}
\int_{-4}^0 (-y) - (y^2 + 3y)\, dy &= \int_{-4}^0 -y^2 - 4y\, dy \\
&= \left(\frac{-y^3}{3} - 2y^2\right)\Bigg|_{-4}^0 \\
&= -\left(\frac{-(-4)^3}{3} - 2(-4)^2\right) \\
&= -\left(\frac{64}{3} - 32\right) \\
&= \frac{32}{3}.
\end{aligned}$$

(16) Plot the curve given by the parametric equations $x = 2t - \sin(t)$, $y = 2 - \cos(t)$, $0 \leq t \leq 2\pi$ and set up (but do not evaluate) an integral expressing its length.

First, draw the graphs of $x = \sin(t)$ and $x = 2t$ on the same (x,t) coordinate plane. Notice that the difference $2t - \sin(t)$ is always positive, and that it is increasing as t gets larger (the derivative $2 - \cos(t)$ is strictly positive.) So, the graph of the parametrized curve is actually the graph of a function since it will satisfy the vertical line rule. Now draw the graphs of $y = \cos(t)$ and $y = 2$ together on a (y,t) coordinate plane. Notice that the difference $2 - \cos(t)$ is always positive, but that it oscillates, starting at 1, increasing to 3, and then going back to 1. The curve is thus a function with a "hump" in the middle:

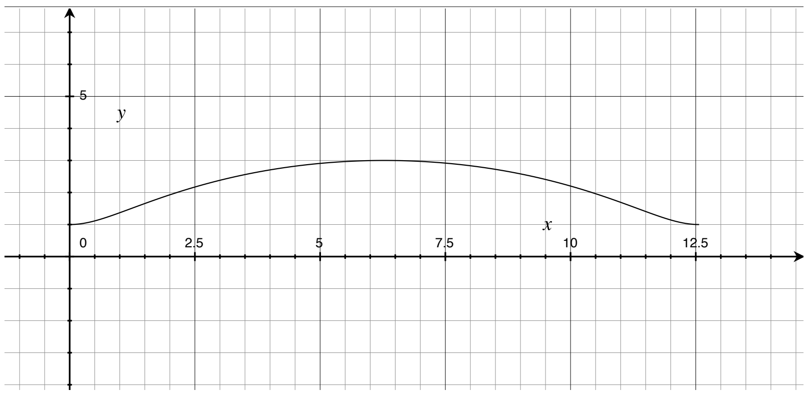


FIGURE 1. The parameterized curve

To find the arc length, we compute $\frac{dx}{dt} = 2 - \cos(t)$ and $\frac{dy}{dt} = \sin(t)$. So, the arc length is given by the integral

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(2 - \cos(t))^2 + (\sin(t))^2}\, dt.$$

(17) Consider the region $\mathcal{R}$ bounded by the lines $y = x + 1$, $x = 0.5$, and $y = 1$. Set up integrals representing the volume of the solid obtained by rotating $\mathcal{R}$ about the x-axis and about the y-axis. Can you find two different integrals for the rotation about each axis?

Using washers to rotate about the x-axis, we get

$$V = \int_0^{1/2} \pi(1 + x)^2 - \pi\, dx.$$

Using shells about the y-axis, we get

$$V = \int_0^{1/2} 2\pi x((1 + x) - 1)\, dx = \int_0^{1/2} 2\pi x^2\, dx$$

Now the "awkward" way: In terms of y, the line $y = x+1$ is $x = y-1$, and its intersection with $x = 1/2$ occurs at $y = 3/2$. So rotating about the y-axis using washers, we get

$$V = \int_1^{3/2} \pi(1/2)^2 - \pi(y - 1)^2\, dy.$$

Rotating about the x-axis and using shells, we get

$$V = \int_1^{3/2} 2\pi y(1/2 - (y - 1))\, dy.$$

(18) A *paraboloid of revolution* is the shape obtained by rotating a parabola of the form $y = ax^2$, (where as a constant) about the y axis. A tank full of water has the shape of a paraboloid of revolution, generated by the curve $y = x^2/2$ and has height $18ft$. (Assume that the density of the water is $62.5lb/ft^3$)

(a) Find the work required to pump the water out of the tank.
I am going to assume we are pumping the water out of the top of the tank.
The radius of the tank at height y is $r(y) = \sqrt{2y}$. Thus, the area A at height y is $A(y) = \pi(r(y))^2 = 2\pi y$. Thus, the infinitesimal volume at height y is given by

$$dV = 2\pi y\, dy.$$

We're using the silly American unit "lb" which already incorporate the acceleration due to gravity, so F is the force density 62.5 times volume V, so

$$dF = 62.5 \cdot dV = 62.5 \cdot 2\pi y\, dy.$$

Finally, work W is distance times force. The distance we need to move a slice at height y is $18 - y$, so the infinitesimal work dW is given by

$$dW = (18 - y)dF = (18 - y)62.5 \cdot 2\pi y\, dy.$$

Therefore, the total work is given by

$$\begin{aligned}
W &= \int_0^{18} dW \\
&= \int_0^{18} (18 - y)62.5 \cdot 2\pi y\, dy \\
&= 62.5 \cdot 2\pi \int_0^{18} (18y - y^2)\, dy \\
&= 125\pi \left(9y^2 - \frac{y^3}{3}\right)\Bigg|_0^{18} \\
&= 125\pi \left(9(18)^2 - \frac{18^3}{3}\right) \\
&= 125\pi \cdot 972.
\end{aligned}$$

(b) After 5000 ft-lb work has been done, what is the depth of the water remaining in the tank?
We find the required depth d by solving the equation

$$5000 = \int_d^{18} (18 - y)62.5 \cdot 2\pi y\, dy.$$

Note that the bounds are d to 18, not 0 to d. Simplifying, this becomes

$$\begin{aligned}
5000 &= 62.5 \cdot 2\pi \int_d^{18} (18 - y)y\, dy \\
&= 125\pi \left(9y^2 - \frac{y^3}{3}\right)\Bigg|_d^{18} \\
&= 125\pi \left(972 - \left(9d^2 - \frac{d^3}{3}\right)\right).
\end{aligned}$$

Simplifying, we have the following cubic polynomial:

$$-\frac{1}{3}d^3 + 9d^2 + \frac{40}{\pi} - 972 = 0.$$

I'm not about to solve for the roots of a cubic polynomial, and neither will you be expected to do so on your final. I just used a calculator and got a reasonable root $d \simeq 16.8$ This makes sense. The top of the tank has a radius of 6 ft. The top foot of water in the tank weighs roughly 7000 lbs. Water is very heavy.

(19) A force of 20N is required to maintain a spring stretched from its natural length of 12cm to a length of 15cm. Find how much work is done in stretching the spring from 12cm to 20cm.

We need to find the spring constant k used in Hooke's law: we have $20 = k(15-12)$, so $k = 20/3$. Thus, the work required to stretch the spring from 12cm to 20cm is

$$\int_0^8 \frac{20}{3}x\,dx = \frac{640}{3},$$

the units being N-cm.

(20) For the differential equation $y' = (y-1)(y-3)(y-5)$,

(a) Sketch a direction field.

First, notice that the direction field will be independent of x. The slope only depends on the height y. The function $(y-1)(y-3)(y-5)$ is a cubic which has been factored for us, with roots at $y=1$, $y=3$ and $y=5$. So the slope is zero for those values, negative below $y=1$, positive between $y=1$ and $y=3$, negative between $y=3$ and $y=5$, and positive again above $y=5$.

(b) Sketch the graphs of the solutions with initial conditions $y(0)=3$, $y(0)=4$ and $y(0)=6$.

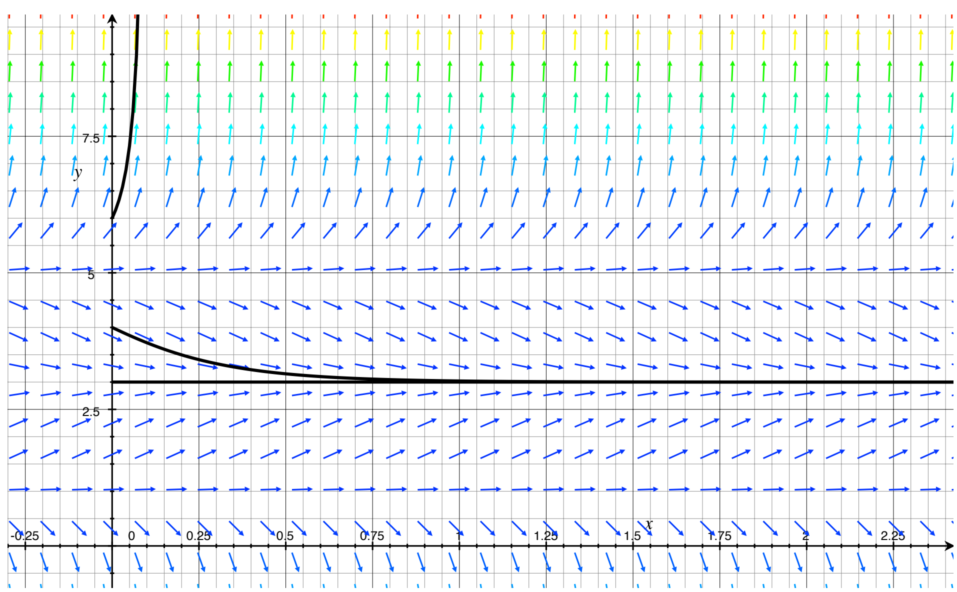


FIGURE 2. The direction field, with the three IVP solutions superimposed. I couldn't get my graphing program to show this, but the IVP solutions should extend in the $-x$ direction, too.

(c) If the initial condition is $y(0) = r$, for which values of r is $\lim_{t\to\infty} y(t)$ finite?
If we start above the value $y(0) = 5$, or below the value $y(0) = 1$, then the limit will not be finite. Otherwise, the graph of $y(t)$ will be trapped between the values $y = 1$ and $y = 5$ since it can never cross those two lines (if it did cross, it would need to do so with non-zero slope, but we know the slope at those levels is zero.)

(21) Sketch the direction field for the differential equation $y' = y - x$. Then use the direction field to sketch four solutions that satisfy the initial conditions $y(0) = 0$, $y(0) = 1$, $y(0) = -3$ and $y(0) = 3$.

To sketch the direction field, notice that the slope y' is zero along the line $y = x$. in the region above that line (where $y > x$,) the slope y' is positive, and below the line it's negative. Along the line $y = x + 1$, the slope $y' = 1$. Any solution starting above this line will go off to infinity, and any solution starting below the line will go to negative infinity.

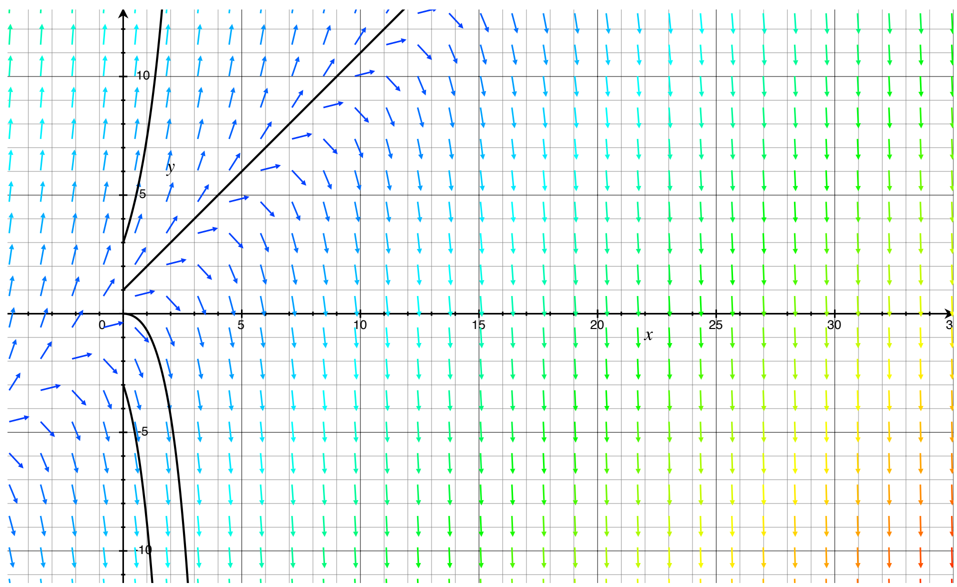


FIGURE 3. The direction field with the solutions superimposed. They should also extend in the $-x$ direction, but my grapher doesn't show this.

(22) Solve the differential equation $3y^2e^{y^3}y' = 4x^3 - 3\sqrt{x}$.

The differential equation is separable. Separating the variables and integrating, we get

$$\int 3y^2e^{y^3}\,dy = \int (4x^3 - 3\sqrt{x})\,dx.$$

The left hand integral is solved using u-substitution: let $u = y^3$. Then $du = 3y^2dy$. So we get

$$\int 3y^2e^{y^3}\,dy = \int e^u\,du = e^u + C = e^{y^3} + C.$$

The right hand integral is solved using the power rule. we get

$$\int (4x^3 - 3\sqrt{x})\,dx = x^4 - 2x^{3/2} + C.$$

Setting the two sides equal (and combining the $+C$'s,) we get

$$e^{y^3} = x^4 - 2x^{3/2} + C.$$

Now we solve for y: take ln of both sides, then take a cube root:

$$y = (\ln(x^4 - 2x^{3/2} + C))^{1/3}.$$

(23) Solve the initial value problem $y' = y(3x + 1)$, $y(0) = 5$.

The differential equation is separable. Separating the variables and integrating, we get

$$\int \frac{dy}{y} = \int (3x + 1)\, dx.$$

The left hand side integrates to $\ln(y) + C$. The right hand side integrates to $\frac{3}{2}x^2 + x + C$. So,

$$\ln(y) = \frac{3}{2}x^2 + x + C,$$

or exponentiating both sides,

$$y = e^{\frac{3}{2}x^2+x+C}.$$

The initial condition $y(0) = 5$ means that $5 = e^{\frac{3}{2}(0)^2+0+C} = e^C$. In other words, $C = \ln(5)$. So the solution to the IVP is

$$y = e^{\frac{3}{2}x^2+x+\ln(5)} = 5e^{\frac{3}{2}x^2+x}.$$

(24) Solve the initial value problem $y'' + 2y' + y = 0$, $y(0) = 5$, $y'(0) = 3$. (note there's a typo in the problem: the $+1$ should be a $+y$.)

We first find the roots of the associated polynomial $r^2 + 2r + 1$. Factoring, we get $(r+1)^2$, so there's one repeated real root at $r = -1$. Thus, the general solution to the differential equation is

$$y = C_1e^{-x} + C_2xe^{-x}.$$

Its derivative is

$$y' = -C_1e^{-x} + C_2e^{-x} - C_2xe^{-x}.$$

The initial condition $y(0) = 5$ means that

$$5 = C_1e^{-0} + C_2(0)e^{-0} = C_1.$$

The initial condition $y'(0) = 3$ means that

$$3 = -C_1e^{-0} + C_2e^{-0} - C_2(0)e^{-0} = -C_1 + C_2.$$

We conclude that $C_1 = 5$ and $C_2 = 3 + C_1 = 8$. Thus the solution to the IVP is

$$y = 5e^{-x} + 8xe^{-x}.$$

(25) Solve the initial value problem $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

The roots of the associated polynomial $r^2 + 4$ are $2i$ and $-2i$. So, the general solution to the differential equation is

$$y = C_1 \cos(2x) + C_2 \sin(2x).$$

Thus,

$$y' = -2C_1 \sin(2x) + 2C_2 \cos(2x).$$

The initial value $y(0) = 1$ means that

$$1 = C_1 \cos(2(0)) + C_2 \sin(2(0)) = C_1.$$

The initial value $y'(0) = 3$ means that

$$3 = -2C_1 \sin(2(0)) + 2C_2 \cos(2(0)) = 2C_2.$$

Thus, $C_1 = 1$ and $C_2 = 3/2$, and the solution to the IVP is

$$y = \cos(2x) + \frac{3}{2}\sin(2x).$$

(26) Solve the initial value problem $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = -1$.

The roots of the associated polynomial $r^2 - 4$ are $r_1 = 2$, $r_2 = -2$. So, the general solution to the differential equation is

$$y = C_1 e^{2x} + C_2 e^{-2x}.$$

Thus,

$$y' = 2C_1 e^{2x} - 2C_2 e^{-2x}.$$

The initial value $y(0) = 1$ means that

$$1 = C_1 e^{2(0)} + C_2 e^{-2(0)} = C_1 + C_2.$$

The initial value $y'(0) = -1$ means that

$$-1 = 2C_1 e^{2(0)} - 2C_2 e^{-2(0)} = 2C_1 - 2C_2$$

. So, we have the system of equations

$$\begin{aligned} C_1 + C_2 &= 1 \\ 2C_1 - 2C_2 &= -1. \end{aligned}$$

Multiply the top row by 2 and add it to the second row. We get

$$4C_1 = 1,$$

So $C_1 = 1/4$ and $C_2 = 3/4$, and the solution to the IVP is

$$y = \frac{1}{4}e^{2x} + \frac{3}{4}e^{-2x}.$$

(27) The state game commission releases 100 dear into a game preserve. During the first 5 years the population increases to 450 deer. Find a model for the population growth assuming logistic growth with a limit of 5000 deer. What does the model predict the size of the population will be in 10 years, 20 years, 30 years?

The logistic model is governed by the differential equation

$$P' = kP(1 - \frac{P}{5000}),$$

where $P(t)$ is the population after t year, and k is the constant of proportionality. The general solution (which we solved for several times in class) is

$$P(t) = \frac{5000P_0 e^{kt}}{5000 + P_0(e^{kt} - 1)}.$$

Here, the initial population P_0 is 100. We want to solve for k. We know that

$$450 = P(5) = \frac{5000 \cdot 100e^{5k}}{5000 + 100(e^{5k} - 1)}.$$

Solving for k, we get

$$k = \frac{1}{5}\ln\frac{441}{91}.$$

Thus,

$$P(t) = \frac{5000 \cdot 100e^{\frac{t}{5}\ln\frac{441}{91}}}{5000 + 100(e^{\frac{t}{5}\ln\frac{441}{91}} - 1)} = \frac{5000(\frac{441}{91})^{t/5}}{49 + (\frac{441}{91})^{t/5}}.$$

In 10 years, the population should be

$$P(10) = \frac{5000(\frac{441}{91})^2}{49 + (\frac{441}{91})^2} = 1620.$$

Likewise, $P(20) \simeq 4592$ and $P(30) \simeq 4981$.

(28) (a) Use Euler's method with step size 0.2 to estimate $y(0.4)$ where $y(t)$ is the solution of the initial value problem $y' = 2.t.y^2$, $y(0) = 1$.
I'm going to assume the differential equation is $y' = 2ty^2$. I'm not sure if what's written is a typo. We have $t_0 = 0$, $t_1 = 0.2$, and $t_2 = 0.4$. We also know $y_0 = 1$, and we're trying to find y_2. Euler's method gives

$$\begin{aligned} y_1 &= y_0 + 0.2(2 \cdot t_0 \cdot (y_0)^2) = 1 + 0.2(2 \cdot 0 \cdot (1)^2) = 1, \\ y_2 &= y_1 + 0.2(2 \cdot t_1 \cdot (y_1)^2) = 1 + 0.2(2 \cdot 0.2 \cdot (1)^2) = 1.08. \end{aligned}$$

So, $y(0.4) \simeq 1.08$.

(b) Find the exact solution of the differential equation and compare the value at 0.4 with the approximation in part a.
Separating the differential equation and integrating, we get

$$\int \frac{dy}{y^2} = \int 2t\,dt.$$

Integrating, we get

$$\frac{-1}{y} = t^2 + C,$$

and solving for y, we get

$$y = \frac{-1}{t^2 + C}.$$

The initial value $1 = y(0) = -1/C$ implies that $C = -1$, so the exact solution to the IVP is

$$y = \frac{-1}{t^2 - 1}.$$

The exact value at $t = 0.4$ is $\frac{-1}{(0.4)^2 - 1} \simeq 1.19$, which is close to our approximation 1.08.

(29) Find a general term for the sequence $\frac{1}{2}, \frac{4}{7}, \frac{1}{2}, \frac{8}{19}, \frac{5}{14}, \frac{4}{13}, \frac{7}{26}, \frac{16}{67}, \frac{3}{14}, \frac{20}{103}, \ldots$ and determine whether it is convergent.

Some of those fractions have been simplified to obfuscate the pattern. Rewrite the sequence as

$$\frac{2}{4}, \frac{4}{7}, \frac{6}{12}, \frac{8}{19}, \frac{10}{28}, \frac{12}{39}, \frac{14}{52}, \frac{16}{67}, \frac{18}{84}, \frac{20}{103}, \ldots.$$

So, the numerator is just $2n$. That leaves the denominator. 103 is suspiciously close to $100 = 10^2$. And 84 is the same distance from 81, which is also a perfect square. In fact, all the denominators are three more than a perfect square. So, we can rewrite the sequence as

$$\frac{2}{3+1^2}, \frac{4}{3+2^2}, \frac{6}{3+3^2}, \frac{8}{3+4^2}, \frac{10}{3+5^2}, \frac{12}{3+6^2}, \frac{14}{3+7^2}, \frac{16}{3+8^2}, \frac{18}{3+9^2}, \frac{20}{3+10^2}, \ldots$$

. So, the general term of the sequence is $a_n = \frac{2n}{3+n^2}$. To see if the sequence is convergent, we compute the limit using L'Hôpital's rule:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{2n}{3+n^2} \\ &= \lim_{n\to\infty} \frac{2}{2n} \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0.\end{aligned}$$

So, the sequence is convergent, and it converges to zero.

(30) Determine whether the series is convergent or divergent. If you are using a test, name it and explain why you can use it.

(a) $\sum_{n=1}^{\infty} \frac{n+1}{n^4+2}$.
Use the limit comparison test, comparing it to the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges by the p-series test, since $p = 3$:

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{\frac{1}{n^3}}{\frac{n+1}{n^4+2}} \right| &= \lim_{n\to\infty} \left| \frac{n^4+2}{n^4+n^3} \right| \\ &= \lim_{n\to\infty} \left| \frac{1+2/n^4}{1+1/n} \right| \\ &= 1.\end{aligned}$$

Since $0 < 1 < \infty$, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^4+2}$ converges too.

(b) $\sum_{n=1}^{\infty} \frac{n^2}{n+2}$.
By L'Hôpital's rule, the limit of the sequence $a_n = \frac{n^2}{n+2}$ is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n+2} = \lim_{n\to\infty} \frac{2n}{1} = \infty.$$

Therefore, since the sequence a_n does not converge to zero, the series $\sum_{n=1}^{\infty} \frac{n^2}{n+2}$ diverges (they name this test in the book, but I forget what it's called.)

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1/3}}$.
The series converges by the alternating series test, since $0 \le \frac{1}{n^{1/3}}$ and $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$.

(d) $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$.
The series is not well-defined, since the $n = 1$ term is $1/0$. Let's assume they mean $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. It diverges by the integral test:

$$\begin{aligned}\int_2^{\infty} \frac{1}{x\ln(x)}\,dx &= \lim_{t\to\infty} \int_2^{t} \frac{1}{x\ln(x)}\,dx \\ &= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{du}{u} \\ &= \lim_{t\to\infty} \ln(u)\Big|_{\ln(2)}^{\ln(t)} \\ &= \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(2)) \\ &= \infty,\end{aligned}$$

where we made the substitution $u = \ln(t)$ in the 2nd line.

(e) $\sum_{n=1}^{\infty} \frac{7^{3n}}{n^2 10^n}$.
Since $7^3/10 = 34.3$, we can rewrite this series as $\sum_{n=1}^{\infty} \frac{34.3^n}{n^2}$. Let $a_n = \frac{34.3^n}{n^2}$. Applying L'Hôpital's rule twice, we compute $\lim_{n\to\infty} a_n$:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{34.3^n}{n^2} \\
&= \lim_{n\to\infty} \frac{\ln(34.3)\cdot(34.3)^n}{2n} \\
&= \frac{\ln(34.3)}{2} \lim_{n\to\infty} \frac{(34.3)^n}{n} \\
&= \frac{\ln(34.3)}{2} \lim_{n\to\infty} \frac{\ln(34.3)\cdot(34.3)^n}{1} \\
&= \frac{(\ln(34.3))^2}{2} \lim_{n\to\infty} (34.3)^n \\
&= \infty.
\end{aligned}$$

Therefore since $\lim_{n\to\infty} a_n \neq 0$, the series does not converge.

(31) Determine the values of x for which the series $\sum_{n=1}^{\infty} e^{nx}$ converges.
The series $\sum_{n=1}^{\infty} e^{nx}$ is geometric, since we can rewrite it as

$$\sum_{n=1}^{\infty} (e^x)^n.$$

Therefore, the series converges if and only if $e^x < 1$, or in other words, if $x < 0$.

(32) Determine how many terms in the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^4}$ are enough to add to approximate the sum correct up to 3 decimal places.
The error E_N is the remainder $E_N = \sum_{n=N+1}^{\infty} (-1)^n \frac{1}{n^4}$. Since the series is alternating, we know that

$$|E_N| \leq |a_{N+1}| = \frac{1}{(N+1)^4}.$$

So, we want to find N such that

$$\frac{1}{(N+1)^4} < 0.001.$$

In other words,

$$1000^(1/4) - 1 < N.$$

The left hand side is approximately 4.62, so we need $N \geq 5$..

(33) Find the radius of convergence and the interval of convergence of the series.
(a) $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$.
Let's apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^4 4^{n+1}} \cdot \frac{n^4 4^n}{x^n} \right| &= \frac{|x|}{4} \lim_{n\to\infty} \left| \frac{n^4}{(n+1)^4} \right| \\
&= \frac{|x|}{4},
\end{aligned}$$

where the limit is computed using L'Hôpital's rule. Thus, the series converges if $|x|/4 < 1$, or if $|x| < 4$. Thus, the radius of convergence is 4. To find the interval of convergence, we need to determine what happens at $x = 4$ and $x = -4$. If $x = 4$, then the series is just $\sum_{n=1}^{\infty} \frac{1}{n^4}$ which converges by the p-test, $p = 4$. if $x = -4$, then the series is just

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$, which converges by the alternating series test. Therefore, the interval of convergence is $[-4, 4]$.

(b) $\sum_{n=1}^{\infty} \frac{(x+10)^n}{n!}$.

Let's apply the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x+10)^{n+1}}{(n+1)!}\cdot\frac{n!}{(x+10)^n}\right| &= |x+10|\lim_{n\to\infty}\frac{1}{n+1}\\ &= |x+10|\cdot 0\\ &= 0.\end{aligned}$$

Therefore, the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

(c) $\sum_{n=1}^{\infty}(-1)^n \frac{(x-1)^n}{n^{1/3}}$.

Let's apply the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)^{1/3}}\cdot\frac{n^{1/3}}{(-1)^n(x-1)^n}\right| &= |x-1|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/3}\\ &= |x-1|.\end{aligned}$$

Thus, the series converges if $|x-1| < 1$. So, the radius of convergence is 1. To find the interval of convergence, we need to find out what happens at $x = 2$ and $x = 0$. If $x = 2$, then the series is just $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$, which converges by the alternating series test. If $x = 0$, then the series is

$$\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^{1/3}} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^{1/3}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/3}},$$

which diverges by the p-series test, since $p = 1/3$. So, the interval of convergence is $(0, 2]$.

(34) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(3n)!x^n}{(n!)^3}$.

We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(3(n+1))!x^{n+1}}{((n+1)!)^3}\cdot\frac{(n!)^3}{(3n)!x^n}\right| &= \lim_{n\to\infty}\left|\frac{(3n+3)!xn!n!n!}{(n+1)!(n+1)!(n+1)!(3n)!}\right|\\ &= \lim_{n\to\infty}\left|\frac{(3n+3)(3n+2)(3n+1)x}{(n+1)^3}\right|\\ &= |x|\lim_{n\to\infty}\frac{27n^3+54n^2+30n+3}{n^3+3n^2+3n+1}\\ &= |x|\cdot 27,\end{aligned}$$

where the last line follows from three applications of L'Hôpital's rule. Thus, the series converges if $|x| \cdot 27 < 1$, so the radius of convergence is $1/27$.

(35) Find the Taylor series of $f(x) = \sin x$ at $x = \pi$.

The Taylor series at $x = a$ of the function $f(x)$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

So, we need to compute derivatives of $f(a) = \sin(a)$:

$$\begin{aligned} f^{(0)}(a) &= \sin a \\ f^{(1)}(a) &= \cos a \\ f^{(2)}(a) &= -\sin a \\ f^{(3)}(a) &= -\cos a, \end{aligned}$$

after which point the pattern repeats. If $a = 0$, we get the pattern $0, 1, 0, -1, \ldots$, but we are instead concerned with what happens at $a = \pi$, in which case the pattern is $0, -1, 0, 1, \ldots$. Notice that this is exactly the negative of the $a = 0$ pattern, so the coefficients in the Taylor series we're trying to find are exactly the negative of the coefficients of the usual MacLaurin series, which is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Thus, the Taylor series at $x = \pi$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1}.$$

(36) Find the MacLaurin series of $f(x) = \frac{x^3}{1+x}$.

We could spend all day trying to use the definition $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, or we can be clever: We know that the geometric series $\sum_{n=0}^{\infty} y^n$ sums to $\frac{1}{1-y}$, as long as $|y| < 1$. Therefore, by plugging in $y = -x$, we get

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Multiplying both sides by x^3 we find the desired series:

$$\frac{x^3}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^{n+3}.$$

(37) Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

(a) Find the interval of convergence.
Let's apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = |x| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |x|,$$

where the limit was computed using L'Hôpital's rule. Thus, the radius of convergence is 1. We need to find out what happens at $x = 1$ and $x = -1$: If $x = 1$, then the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-test, since $p = 2$. If $x = -1$, then the series converges by the alternating series test. So, the interval of convergence is $[-1, 1]$.

(b) Estimate $f(-1/2)$ by adding four terms.

$$f(-1/2) \simeq -\frac{1}{2} + \frac{1/4}{4} - \frac{1/8}{9} + \frac{1/16}{16} \simeq 0.447482$$

.

(c) Determine how many terms of the series $f(-1/2)$ are required to ensure that the sum is accurate to within 0.0001.
The series $f(-1/2)$ is given by $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2}$. It is alternating, so We can use the alternating series error estimate. In the notation of Problem (32), we know that

$$|E_N| \leq \frac{1}{2^{N+1}(N+1)^2}.$$

We want $\frac{1}{2^{N+1}(N+1)^2} \leq 0.0001$, or in other words

$$10,000 \leq 2^{N+1}(N+1)^2.$$

A little trial and error later, and we can find that $N = 7$ works: $2^8 \cdot 8^2 = 16,384 > 10,000$.

(38) Use series to evaluate the limit $\lim_{x\to 0} \frac{x^2 e^x}{\cos x - 1}$.
We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

so

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{3!} + \frac{x^6}{4!} + \dots$$

We also know that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots,$$

so

$$\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots.$$

Therefore,

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2 e^x}{\cos x - 1} &= \lim_{x\to 0} \frac{x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{3!} + \frac{x^6}{4!} + \dots}{-\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots} \\
&= \lim_{x\to 0} \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}{-\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots} \\
&= \frac{\lim_{x\to 0}(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots)}{\lim_{x\to 0}(-\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots)} \\
&= \frac{1}{-1/2} \\
&= -2.
\end{aligned}$$