Practice Final Selected Solutions

Question 11c

$$\int_{-10}^{1} \frac{x}{\sqrt{x+10}} dx \quad (1)$$

$$= \lim_{a \to -10} \int_{-a}^{1} \frac{x}{\sqrt{x+10}} dx \quad (2)$$

$$u = x + 10 \Rightarrow x = u - 10 \quad (3)$$

$$du = dx, ub = 11, lb = 0 \quad (4)$$

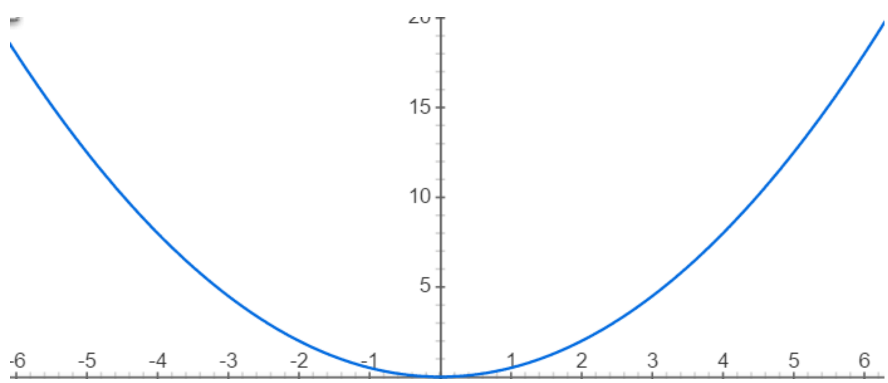
$$= \int_{0}^{11} \frac{u-10}{\sqrt{u}} du \quad (5)$$

$$= \lim_{b \to 0} \left(\int_{b}^{11} \frac{u}{\sqrt{u}} du - 10 \int_{b}^{11} \frac{1}{\sqrt{u}} du \right) \quad (6)$$

$$= \lim_{b \to 0} \left(\frac{2}{3} u^{\frac{3}{2}} \Big|_{b}^{11} - 20u^{\frac{1}{2}} \Big|_{b}^{11} \right) \quad (7)$$

$$= \frac{2}{3} \cdot 11^{\frac{3}{2}} - 20 \cdot 11 \frac{1}{2} \quad (8)$$

Question 18



a) We can rewrite the equation of the graph in terms of y. This results in $x = \sqrt{2y}$.

$$W = \int F dx$$

The force is given as $62.5 \frac{lb}{ft^3}$. The volume is going to be the integral of thin circles of radius r, and thickness dy, where r is the distance from the y-axis to the curve. So, the $V = 62.5\pi(\sqrt{2y})^2 dy$. The height the water has to travel is 18-y, since if the water was at y=0, it would have to travel 18 ft; if the water was 18 ft, it would have to travel 0 ft. Combing all these gives us the expression for the W done.

$$W = \int_0^{18} 62.5\pi(\sqrt{2y})(18 - y)dy$$

$$W = 125\pi(9y^2 - \frac{y^3}{3})\Big|_0^{18}$$

$$W = 125\pi(9 \cdot 18^2 - \frac{18^3}{3})$$

b) In this case you are given the work done and asked to find the height, so we can replace the height given in part, 18, by some arbitrary height h.

$$W = 5000 = \int_0^{h} 62.5\pi(\sqrt{2y})(18 - y)dy$$

Now, you can just treat h as a constant, and evaluate the integral, until you get an algebraic expression (given below) and solve that for h.

$$5000 = 125\pi(9h^2 - \frac{h^3}{3})$$

Question 24

$$y'' + 2y' + y = 0, y(0) = 5, y'(0) = 3 \tag{9}$$

$$\text{Guess:} \quad y(x) = e^{rx}, \text{so} \quad y'(x) = re^{rx}, y''(x) = r^2e^{rx} \tag{10}$$

$$\text{Substitude:} \quad r^2e^{rx} + 2re^{rx} + e^{rx} = 0 \tag{11}$$

$$e^{rx}(r^2 + 2r + 1) = 0 \tag{12}$$

$$e^{rx} \neq 0 \text{ for finite x and r} \quad r^2 + 2r + 1 = 0 \tag{13}$$

$$r_1 = r_2 = -1 \tag{14}$$

$$\text{General Solution:} \quad y(x) = C_1e^{-x} + C_2xe^{-x} \tag{15}$$

$$\text{Using y(0) = 5:} y(0) = 5 = C_1 \tag{16}$$

$$\text{Using y'(0) = 3} y'(0) = 3 = -5 + C_2 \tag{17}$$

$$y(x) = 5e^{-x} + 8xe^{-x} \tag{18}$$

Question 29

$$\frac{1}{2}, \frac{4}{7}, \frac{1}{2}, \frac{8}{19}, \frac{5}{14}, \frac{4}{13}, \frac{7}{26}, \frac{16}{67}, \frac{3}{14}, \frac{20}{103}, \ldots \tag{19}$$

This sequence seem to be increases so we can multiply each fraction by 1 in form of $\frac{x}{x}$, where we pick x, in order to make sure the terms are in ascending order.

$$\frac{1}{2}, \frac{4}{7}, \frac{6}{12}, \frac{8}{19}, \frac{10}{28}, \frac{12}{39}, \frac{14}{52}, \frac{16}{67}, \ldots \tag{20}$$

Now that the sequence looks monotone increases, we can start looking at the differences between the numerators and denominators of each subsequent term. So, notice that the differences in the denominator are as follows: $5, 5, 7, 9, 11, 13, ...$, and the second difference is $0, 2, 2, 2, ...$. Since the second difference is constant, this means that the denominator has an n^2 term in it. In the numerator, the first differences are constant: $3, 2, 2, 2, 2, ...$, which mean it is a linear term n. So, our general term is something like $\frac{n}{n^2}$ Let us ignore the first two terms for now since their difference does not follow the rest of the pattern. For $n = 5$, we get $\frac{5}{25}$, and for $n = 4$, we get $\frac{4}{16}$. In both cases, the denominator is off by 3 and the the numerator is twice as much, so the formula for the sequence is $\frac{2n}{n^2+3}$.

Question 30a

$\sum_{n=1}^{\infty} \frac{2}{n^3} = 2\sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges because of the p-series test as $3 > 1$. $\sum_{n=1}^{\infty} \frac{n+1}{n^4+2} < \sum_{n=1}^{\infty} \frac{n+1}{n^4} < \sum_{n=1}^{\infty} \frac{n+n}{n^4} = \sum_{n=1}^{\infty} \frac{2n}{n^4} = \sum_{n=1}^{\infty} \frac{2}{n^3}$. Since $\sum_{n=1}^{\infty} \frac{2}{n^3}$ converges and $\sum_{n=1}^{\infty} \frac{n+1}{n^4+2} < \sum_{n=1}^{\infty} \frac{2}{n^3}$, $\sum_{n=1}^{\infty} \frac{n+1}{n^4+2}$ must also converge by the comparison text.