(1) The region bounded by the curves $y = x^2$ and $x = y^2$ is rotated about the x-axis. Set up an integral for the volume of the resulting solid by two different methods.

Method 1: Using washers, we have two functions of x: $y = x^2$ and $y = \sqrt{x}$. To find their intersection, we set them equal to each other.

$$\begin{aligned} x^2 &= \sqrt{x} \\ x^4 &= x \\ x^4 - x &= 0 \\ x(x^3 - 1) &= 0 \\ x(x-1)(x^2 + x + 1) &= 0. \end{aligned}$$

By the quadratic formula, the quantity $x^2 + x + 1$ is never zero, so the intersection points are just $x = 0$ and $x = 1$. On the interval $[0, 1]$ the function $y = \sqrt{x}$ is larger than the function $y = x^2$, so the washer method gives

$$A = \int_0^1 \pi(\sqrt{x})^2 - \pi(x^2)^2 dx.$$

Method 2: Using cylindrical shells, we need to write the two curves as functions of y: $x = y^2$ and $x = \sqrt{y}$. The intersection point is found as before, giving us $y = 0$ and $y = 1$. The washer methods gives

$$A = \int_0^1 2\pi y(\sqrt{y} - y^2) dy.$$

(2) Set up (but do not evaluate) an integral for the length of the curve $y = x^{3/2}$, for $x \in [0, 4]$.

We parameterize the curve by $x = t$, $y = t^{3/2}$, $t \in [0, 4]$. Taking derivatives, we get

$$\begin{aligned} \frac{dx}{dt} &= 1; \\ \frac{dy}{dt} &= \frac{3}{2}\sqrt{t}. \end{aligned}$$

Thus, the formula for arc length gives

$$L = \int_0^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{t}\right)^2}\, dt.$$

(3) Set up integral for the length of and the area of the inside loop of the polar curve $r = 1 - 2\cos(\theta)$.

By setting $r = 0$ we see that the curve passes through 0 at $\theta = \pi/3$ and $-\pi/3$. By plotting the graph (plot some points like $\theta = 0, \pm\pi/6, \pm\pi/4, \pm\pi/3, \pm\pi/4$, etc) we see that the loop occurs on the interval $\theta \in [-\pi/3, \pi/3]$. We also need to compute $\frac{dr}{d\theta}$:

$$\frac{dr}{d\theta} = 2\sin(\theta).$$

The formulas for polar arc length L and area A give

$$\begin{aligned} L &= \int_{-\pi/3}^{\pi/3} \sqrt{(1 - 2\cos(\theta))^2 + (2\sin(\theta))^2}\, d\theta \\ A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(1 - 2\cos(\theta))^2\, d\theta. \end{aligned}$$

(4) A tank on the shape of a sphere of radius $10ft$ is full of oil weighing 50lb/ft^3. How much work is done by pumping the oil through a hole in the top?

Placing the origin at the center of the tank, we find that the radius of a slice at height y is given by $r(y) = \sqrt{10^2 - y^2}$. So, the area of a slice is $A(y) = \pi(100 - y^2)$. So, the infinitesimal volume of a slice, dV is given by

$$dV = \pi(100 - y^2)dy.$$

Since mass M is density $\times$ volume, the infinitesimal mass dM is given by

$$dM = 50 \cdot dV = 50 \cdot \pi(100 - y^2)dy.$$

Force F is mass times acceleration, so the infinitesimal force dF required to lift a slice is given by

$$dF = 32 \cdot dM = 32 \cdot 50 \cdot \pi(100 - y^2)dy,$$

where the acceleration due to gravity is roughly 32ft/s^2. We need to move a given slice at height y to the top of the tank, which is at height 10. This distance is $10 - y$. Work is distance times force, so the infinitesimal work for a given slice at height y is

$$dW = (10 - y) \cdot dF = (10 - y) \cdot 32 \cdot 50 \cdot \pi(100 - y^2)dy.$$

The total work W is thus the integral of dW, from $y = -10$ to $y = 10$:

$$W = \int_{-10}^{10} (10 - y) \cdot 32 \cdot 50 \cdot \pi(100 - y^2)dy.$$

You can then integrate this function (first pull out all those constants!) to get the answer.

(5) if 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12cm to 14cm, what is the natural length of the spring? (Recall Hooke's law: the force required to maintain a spring stretched x units beyond its natural length is proportional to x).

In mathematical notation, Hooke's law says that the force $F(x)$ as a function of x is given by the function kx, where k is some constant ($k > 0$.) Let L denote the natural length of the spring. The first statement tells us that

$$\int_{0.1-L}^{0.12-L} kx\,dx = 6$$

The limits of integration are written like that, since 10cm is equal to L plus the initial amount that the spring is being stretched. Likewise 12cm is equal to L plus the final amount being stretched. We needed to change centimeters into meters because Joules are in terms of meters.)

The second statement likewise says

$$\int_{0.12-L}^{0.14-L} kx\,dx = 10.$$

Integrating, we get the system of equations

$$\frac{k}{2}\left[(0.12 - L)^2 - (0.1 - L)^2\right] = 6$$
$$\frac{k}{2}\left[(0.14 - L)^2 - (0.12 - L)^2\right] = 10.$$

Simplifying, we get

$$0.11 - L = \frac{300}{k}$$
$$0.13 - L = \frac{500}{k}.$$

Thus, $k = 300/(0.11 - L)$, so

$$L = 0.13 - \frac{500}{300}(0.11 - L).$$

Solving for L, we get $L = 0.08$ meters, or 8 cm.

(6) Determine whether each of the series is convergent or divergent. If it is convergent, find its sum. (Justify your answers)

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$. Diverges. Use the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which we know diverges (it's a p-series with $p = 1$.)

$$\begin{aligned}
\lim_{n\to\infty} \frac{1/n}{1/\sqrt{n^2+1}} &= \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} \\
&= \lim_{n\to\infty} \sqrt{\frac{n^2+1}{n^2}} \\
&= \sqrt{\lim_{n\to\infty} \frac{n^2+1}{n^2}} \\
&= \sqrt{\lim_{n\to\infty} 1 + \frac{1}{n^2}} \\
&= \sqrt{1 + \lim_{n\to\infty} \frac{1}{n^2}} \\
&= \sqrt{1+0} \\
&= 1.
\end{aligned}$$

Since $0 < 1 < \infty$, the limit comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges too.

(b) $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{1}{(-10)^n} \right)$. Converges. We can break it up into two series. The first, $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ converges by the comparison test: we compare it to the series $\sum_{n=1}^{\infty} \frac{3}{n^2} = 3\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges since it's a p-series where $p = 2$. We may apply the comparison test, since $0 \le \frac{3}{n(n+3)} \le \frac{3}{n^2}$.

The second series $\sum_{n=1}^{\infty} \frac{1}{(-10)^n}$ converges since it's a geometric series $\sum r^n$, where $r = -1/10$ (hence $|r| < 1$.) It converges to $\frac{1}{1+1/10} - 1$. (we subtract 1, because the formula for a geometric series starts at $n = 0$, and we're starting from $n = 1$.)

(c) $\sum_{n=1}^{\infty} \frac{n^2-4}{2n^2+3}$ diverges, since by L'Hopital's rule,

$$\lim_{n\to\infty} \frac{n^2-4}{2n^2+3} = \lim_{n\to\infty} \frac{2n}{4n} = \frac{1}{2}.$$

Since the sequence $\frac{n^2-4}{2n^2+3}$ does not converge to zero, the series can't possibly converge. (Recall that if a series $\sum a_n$ converges, then $a_n \to 0$.)

(d) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges by the comparison test. We know that for $n \ge 3$ we have $\frac{\ln(n)}{n} > \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (it's a p-series, where $p = 1$,) the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ also diverges.

(7) Represent the number $3.4\overline{15} = 3.4151515\ldots$ as a quotient of integers.

We first represent the number as a series: $3.4\overline{15} = 3.4 + 0.0\overline{15}$. We have

$$\begin{aligned} 0.015 &= \frac{15}{1000} \\ 0.01515 &= \frac{15}{1000} + \frac{15}{1000} \cdot \frac{1}{100} \\ 0.0151515 &= \frac{15}{1000} + \frac{15}{1000} \cdot \frac{1}{100} + \frac{15}{1000} \cdot \frac{1}{100} \cdot \frac{1}{100}. \end{aligned}$$

So, we begin to see a pattern. The repeating decimal is thus given by

$$0.0\overline{15} = \sum_{n=0}^{\infty} \left(\frac{15}{1000}\right) \cdot \frac{1}{100^n} = \frac{15}{1000} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n = \frac{15}{1000} \cdot \frac{1}{1 - \frac{1}{100}},$$

since $\sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n$ is a geometric series $\sum_{n=0} \infty r^n$ where $r = 1/100$. Therefore, the full decimal $3.4\overline{15}$ is given by

$$\begin{aligned} 3.4\overline{15} &= 3 + \frac{4}{10} + 0.0\overline{15} \\ &= \frac{34}{10} + \frac{15}{1000} \cdot \frac{1}{1 - \frac{1}{100}} \\ &= \frac{34}{10} + \frac{15}{10}\frac{1}{99} \\ &= \frac{34 \cdot 99 + 15}{990} \\ &= \frac{3381}{990}. \end{aligned}$$

(8) (a) Approximate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the first 4 terms.

We compute this by hand: the sum of the first 4 terms is $1 + 1/4 + 1/9 + 1/16$.

(b) Estimate the error of the approximation. In general, the error $E_N = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{N} \frac{1}{n^2}$ satisfies

$$\int_{N+1}^{\infty} \frac{1}{x^2} dx \leq E_N \leq \int_{N}^{\infty} \frac{1}{x^2} dx.$$

We integrate both of these improper integrals (remembering to take a limit!) to get

$$\frac{1}{N+1} \leq E_N \leq \frac{1}{N}.$$

Here, $N = 4$, so the error is between $1/5$ and $1/4$.

(c) Determine how many terms are required to ensure that the sum is accurate to within 0.0001.

By part (b), we know that the error $E_N \leq 1/N$. We want to find N so that $1/N < 0.0001$ (so then $E_N \leq 1/N < 0.0001$). Rewriting this as a fraction, we want N so that $1/N < 1/10000$, or in other words $N > 10000$.

Good luck tomorrow!!!