Integrals

"Tricks"
Improper integrals

(1) Evaluate the integral $\int (\ln(x))^3 \, dx$.

(2) Evaluate the integral $\int \cos^4(3x) \, dx$.

(3) Evaluate the integral $\int_0^\pi e^x \sin(\pi - x) \, dx$.

(4) Evaluate the integral $\int \frac{x}{(2x+5)(x-2)} \, dx$.

(5) Evaluate the integral $\int \arctan(1/x) \, dx$.

(6) Evaluate the integral $\int xe^{-3x} \, dx$.

(7) Evaluate the integral $\int \cos^5(3x) \, dx$ and

(8) Prove that the area of a circle with radius r is πr^2.

(9) Evaluate the integral $\int \frac{\sqrt{16-x^2}}{x^2} \, dx$.

(11) Evaluate the integral or show it is divergent

(a) $\int_2^\infty \frac{2}{(2x+3)^4} \, dx$.

(b) $\int_{-\infty}^0 e^{-3x} \, dx$.

(c) $\int_{-10}^1 \frac{x}{\sqrt{x+10}} \, dx$.

(d) $\int_1^e \frac{1}{x \ln \sqrt{x}} \, dx$.

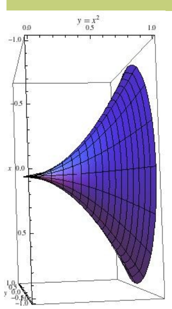
(10) Consider a real number p. Find the values of p for which the integral $\int_2^{\infty} x^p\, dx$ converges, and evaluate the integral for those values of p. .

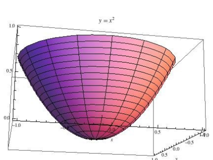
Applications of Integrals

Areas - Volume arc length
(parametric, cartesian and polar coordinates)
Work

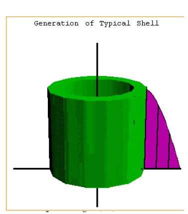
Volumes Summary

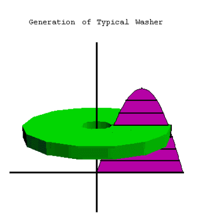
The solids below are obtained by rotating a region of the plane about vertical or horizontal axes. They are called *solids of revolution*.





Rotating about y axis





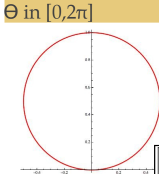
$$V = \int_a^b 2\pi x f(x)\, dx$$

$$\pi \int_a^b (g(y)^2 - f(y)^2) dy$$

We discussed how to find the volume of solids of revolution by the washer method or the cylindrical shell method.

ARC LENGTH (recall Pythagorean Theorem)

Example: Compute the arc length of the circle r = sinϴ
ϴ in [0,2π]

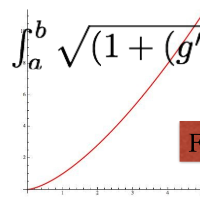


POLAR

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example: Compute the arc length the graph of the curve $y=x^{3/2}$,
x in [0,5]

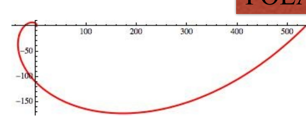
$$\int_a^b \sqrt{(1 + (g'(x))^2}\ dx$$



FUNCTION

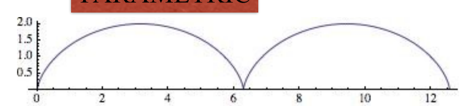
Example: Compute the length arc of the spiral $r = e^{\theta}$
in [0,2π]

POLAR



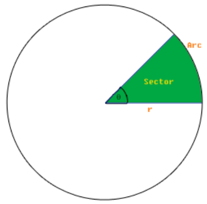
Example: Compute the length two arcs of the cycloid x= t - sin t, y=1-cos t. (Hint (1-cos(t))=2 sin²(t/2))

PARAMETRIC



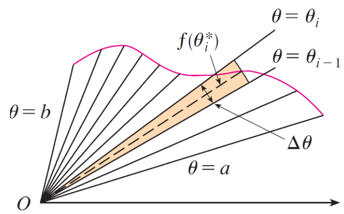
$$\int_c^d \sqrt{((f'(t))^2 + (g'(t))^2}\ dt$$

Area of a sector of a circle of radius r



$$\frac{\theta}{2\pi} = \frac{A}{\pi r^2} \Rightarrow A = \frac{r^2\theta}{2}$$

Area A of a region bounded by a polar curve of equation r=f(Θ), Θ in [a,b]



$$\Delta\theta r^2/2 = \Delta\theta . f(\theta)^2/2$$

$$A = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta$$

(12) Find the length of the polar curve $r = e^{2\theta}$, $0 \le \theta \le 2\pi$.

(13) Find the area of the region that lies inside the curves $r = \cos(3t)$ and $r = \sin(3t)$.

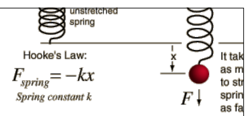
(14) Find the area of the region bounded by the given curves $y = 1/x$, $y = x^3$, $y = 0$ and $x = 2$.

(15) Find the area of the region bounded by the given curves $x + y = 0$ and $x = y^2 + 3y$.

(16) Plot the curve given by the parametric equations $x = 2t - \sin(t)$, $y = 2 - \cos(t)$, $0 \le t \le 2\pi$ and set up (but do not evaluate) an integral expressing its length.

(17) Consider the region $\mathcal{R}$ bounded by the curves $y = x + 1$, $x = 0.5$ and $y = 1$. Set up integrals representing the volume of the solid obtained by rotating $\mathcal{R}$ about the x axis and about the y. Can you find two different integrals for each axis?

EXAMPLE



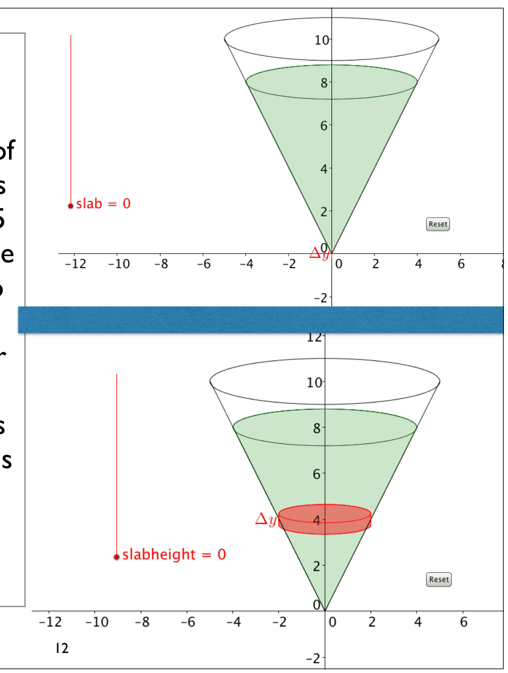
- A spring can by compressed by 4 in. from its natural length of 10in. when a force of 8 lb. is applied. How much work is done in compressing the spring this distance?

Hooke's Law

$f(x) = k . x$

The force required to hold a spring stretched (or compressed) x in. (or other units) beyond its natural length is f(x) = k . x where k is a constant that depends on the spring and on the units of measurement.

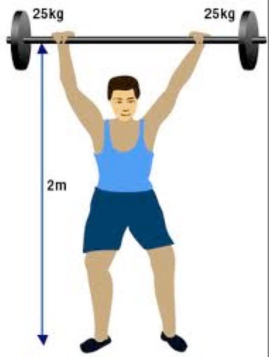
A water tank in the form of an inverted circular cone is 20 ft across the top and 15 ft. deep. If the surface of the water is 5 ft below the top of the talk, find the work done in pumping the water to the top of the thank. (Leave your result in terms of w, the number of pounds per1 ft³ of water)



Computing work

- Find a “wise” location of the x-axis and the origin. Divide into n “pieces”. These “pieces” should be such that one can approximate the work for each of them, with the formula W=F.d.
- Add the work of each piece. This yields a Riemann sum, and then passing to the limit, an integral.

Recall
work = force x distance



(18) A *paraboloid of revolution* is the shape obtained by rotating a parabola of the form $y = ax^2$, (where a is a constant) about the y axis. A tank full of water has the shape of a paraboloid of revolution, generated by the curve $y = x^2/2$ and has height $18ft$.

(a) Find the work required to pump the water out of the tank.

(b) After 5000 ft-lb work has been done, what is the depth of the water remaining in the tank.

(19) A force of 20N is required to mantain a spring stretched from its natural length of 12cm to a length of 15cm. Find how much work is done in stretching the spring from 12cm to 20cm.

Density of water is 62.5 lb/ft^3

Sequences and Series

What are they?
Convergency tests
Power series (Taylor, McLauren, 1/(1+x))

Examples to remember:

- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if p>1.
- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if p≤1.
- The geometric series $\sum_{n=1}^{\infty} a\, r^n$ is convergent if -1<r<1.
- The geometric series $\sum_{n=1}^{\infty} a\, r^n$ is divergent if r≤-1 or r≥1.

Series Summary: Tests and Important Examples

- Divergence test: If the sequence does not converges to 0, then the corresponding series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\sum_{n=1}^{\infty} a\, r^n$$

The three tests below can be applied to series with positive terms.

- Integral test (to apply this test the corresponding function must be decreasing.)
- Comparison test
- Limit comparison test

- Recall: If an =a1 r^n (a geometric sequence), the corresponding series is called geometric series.
- the geometric series is convergent if and only if |r|<1
- its sum is a/(1-r)

The alternating series test

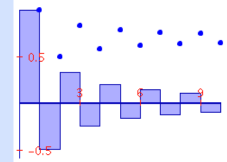
If we have a sequence $\{ a_n \}$,

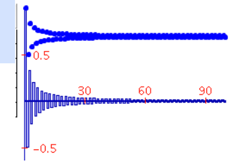
where $a_n > 0$, $a_n \geq a_{n+1}$, and $a_n \to 0$ when $n \to \infty$

then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges





Alternating Series Estimation Theorem If $s = \Sigma (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \lim_{n\to\infty} b_n = 0$$

then $$|R_n| = |s - s_n| \leqslant b_{n+1}$$

Differentiating and integrating power series

2 Theorem If the power series $\Sigma\, c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

$$\text{(i)}\ f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\text{(ii)}\ \int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

- Important example

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

- Find a power series in terms of another one you know.

Examples
$1/(1+8x^3)$, $x^2/(x+5)$

Examples
f(x)=arctan(x)f(x)=ln(x)

- Use the power series corresponding to f′(x) to find the power series corresponding to f(x).

- Use the power series of ∫f(x)dx to find the power series of f(x)

Examples $f(x)=1/(1+x)^2$

Example: Express $1/(1+x)^2$ as a power series and find radius of convergence?.

- Recall that the derivative and integral of a power series have the same interval of convergence.

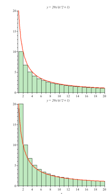
- Use the alternate series estimation test to estimate the error when approximating a series by a partial sum.

Find the value of a certain integral below correct up to 8 decimal places

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

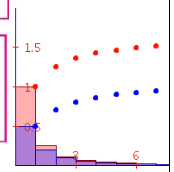
(b) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.



The Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(a) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(b) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.



The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

The series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is called the Taylor series of f at a.

When a=0, the series (below) is called the Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Alternating Series Estimation Theorem If $s = \Sigma\,(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \lim_{n\to\infty} b_n = 0$$

then $$|R_n| = |s - s_n| \leqslant b_{n+1}$$

3 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_n^{\infty} f(x)\,dx$$

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leqslant M$ for $|x-a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leqslant d$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+2)}{n(n+1)} \qquad \sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n$$

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

37–44 Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (definition of a Maclaurin series) or known series such as geometric series, binomial series, or the Maclaurin series for e^x, $\sin x$, and $\tan^{-1}x$.

37. $f(x) = \dfrac{x^2}{1+x}$

38. $f(x) = \tan^{-1}(x^2)$

39. $f(x) = \ln(4 - x)$

40. $f(x) = xe^{2x}$

41. $f(x) = \sin(x^4)$

42. $f(x) = 10^x$

46. Use series to approximate $\int_0^1 \sqrt{1 + x^4}\, dx$ correct to two decimal places.

49. Use series to evaluate the following limit.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

(34) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(3n)!x^n}{(n!)^3}$.

(35) Find the Taylor series of $f(x) = \sin x$ at $x = \pi$.

(36) Find the McLaurin series of $f(x) = \frac{x^3}{1+x}$.

(37) Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

(a) Find the interval of convergence.

(b) Estimate $f(-1/2)$ by adding four terms.

(c) Determine how many terms of the series $f(-1/2)$ are required to ensure that the sum is accurate to within 0.0001.

(38) Use series to evaluate the limit $\lim_{x\to 0} \frac{x^2e^x}{\cos x - 1}$

(31) Determines which values of x does the series $\sum_{n=1}^{\infty} e^{nx}$ converges.

(32) Find the sum of the seris $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^4}$ correct up to 3 decimal places

(33) Find the radius of convergence and the interval of convergence of t series.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$.

(b) $\sum_{n=1}^{\infty} \frac{(x+10)^n}{n!}$.

(c) $\sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^n}{n^{1/3}}$.

(34) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(3n)!x^n}{(n!)^3}$.

(35) Find the Taylor series of $f(x) = \sin x$ at $x = \pi$.

(36) Find the McLaurin series of $f(x) = \frac{x^3}{1+x}$.

29) Find a general term for the sequence $\frac{1}{2}, \frac{4}{7}, \frac{1}{2}, \frac{8}{19}, \frac{5}{14}, \frac{4}{13}, \frac{7}{26}, \frac{16}{67}, \frac{3}{14}, \frac{20}{103}, \ldots$ and determie wether it is convergent.

30) Determine whether the series is convergent or divergent. If you are using a test, name it and explain why you can use it.

(a) $\sum_{n=1}^{\infty} \frac{n+1}{n^4+2}$.

(b) $\sum_{n=1}^{\infty} \frac{n^2}{n+2}$.

(c) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{1/3}}$.

(d) $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$.

(e) $\sum_{n=1}^{\infty} \frac{7^{3n}}{n^2 10^n}$.

Differential Equations

What are they?
Initial value problems
Separable equations
Direction fields
The logistic equation
Second order diff. equation with constant coefficients

Summary on solving the linear second order homogeneous differential equation

To find the general solution of the differential equation **A** y″ + **B** y′ + **C** y = 0 we consider the characteristic equation:

A x^2 + **B** x + **C** = 0

Set Δ = $\mathbf{B}^2$ - 4 **A** **C.**

Example

1. Solve the initial-value problem y″ + 2 y′ + y=0, y(0)=1, y(1)=3.
2. 2y″+5y+3y=0, y(0)=3, y′(0)=-4.

	roots of the characteristic polynomial	General solution
Δ>0	two distinct real roots r	c_1
Δ<0	two complex roots α+iβ and α - iβ	e $^{\alpha}$ sin(βx))
Δ=0	one double real root r	c_1

(20) For the differential equation $y' = (y-1)(y-3)(y-5)$.

(a) Sketch a direction field.

(b) Sketch the graph of the solution with initial condition $y(0) = 3$, $y(0) = 4$ and $y(0) = 6$).

(c) If the initial condition is $y(0) = r$, for which values of r is $\lim_{t\to\infty} y(t)$ is finite?

(21) Sketch the direction field for the differential equation $y' = y - x$. Then use the direction field to schetch four solutions that satisfy the initial conditions $y(0) = 0, y(0) = 1, y(0) = -3$ and $y(0) = 3$.

(22) Solve the differential equation $3y^2 e^{y^3} y' = 4x^3 - 3\sqrt{x}$.

(23) Solve the initial value problem $y' = y(3x+1)$, $y(0) = 5$.

(24) Solve the initial value problem $y'' + 2y' + 1 = 0$, $y(0) = 5$, $y'(0) = 3$.

(25) Solve the initial value problem $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

(26) Solve the initial value problem $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = -1$.

(27) The state game commission releases 100 deer into a game preserve. During the first 5 years the population increases to 450 deer. Find a model for the population growth assuming logistic growth with a limit of 5000 deer. What does the model predict the size of the population will be in 10 years, 20 years, 30 years?

(28) (a) Use Euler's method with step size 0.2 to estimate $y(0.4)$ where $y(t)$ is the solution of the initial value problem $y' = 2.t.y^2$, $y(0) = 1$.

(b) Find the exact solution of the differential equation and comparate the value at 0.4 with the approximation in part a.

9. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.

(a) Write a differential equation that is satisfied by y.

(b) Solve the differential equation.

(c) A small town has 1000 inhabitants. At 8 AM, 80 people have heard a rumor. By noon half the town has heard it. At what time will 90% of the population have heard the rumor?