MAT 132

8.6 Representation of functions as power series

Recall

A **function f** from a set X to a set Y is a rule that assigns to each element of a set X a unique element of a set Y.

Examples:

- X=**Z**, Y=**Z** and f : x → 2·x
- X=**N**, Y=**R** and f : x → 2·n/(n^2+1)
- X=**R**, Y=**R** and f : x → x^2+x+3
- X=**(-π/2,π/2)** , Y=**R** and f : x → tan(x)
- X=**{x in R, x ≠1}**, Y=**R** and f : x → 1/(1-x)
- X=**{x in R, x>-3}**, Y=**R** and f : x → √(x+3)

Recall

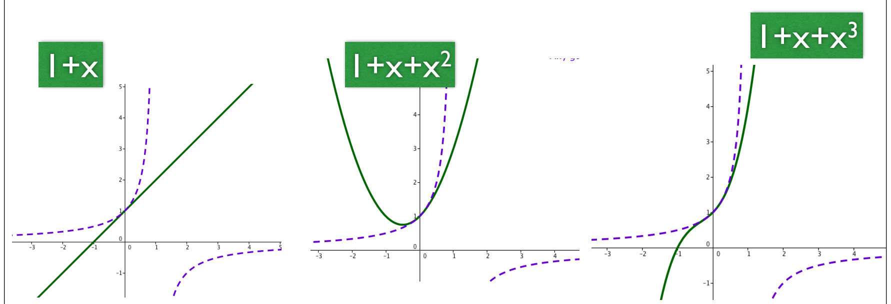
A power series defines a function whose domain is the interval of convergence of the power series.

Example: The geometric series with a_1=1 and r=x

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 \ldots = 1/(1-x)$$

if |x|<1

- Graph the first several partial sums of the geometric series together with the function f(x)=1/(1-x), defined for all x ≠1.
- On what interval do these partial sums appear to be converging to f?



$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

if |x|<1

Examples

1. Express 1/(1+x) as a power series and find the interval of convergence.
2. Express $1/(1+x^3)$ as a power series and find the interval of convergence.
3. Express $1/(1+8x^3)$ as a power series and find the interval of convergence.
4. Find a power series representation for 1/(x+5)
5. Find a power series representation for $x^2/(x+5)$

Differentiating and integrating power series

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

.Consider the power series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2}$

1. Find the radius of convergence.
2. Find the derivative of the function defined by the series.
3. If f is the function defined by the power series, find the domain of f, and the domain of f'. Do f and f' have the same radius of convergence? Do they have the same interval of convergence?

Examples

1. Express $1/(1-x)^2$ as a power series.

What is the radius of convergence?.

2. Find a power series representation of f(x)=arctan(x).
3. Find the value of the integral below correct up to 6 decimal places

$$\int_0^{\frac{1}{5}} \frac{1}{1+x^4}\,dx$$

(1/5)^9/9=1/17578125~5.68889*10^{-8}

$(1/5)^5/5$=1/15625~0.000064

REVIEW The alternating series test

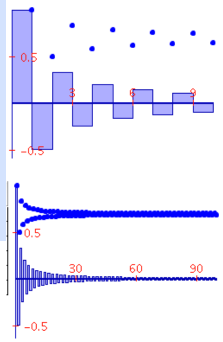
If we have a sequence $\{ a_n \}$,

where $a_n > 0$, $a_n \geq a_{n+1}$, and $a_n \to 0$ when $n \to \infty$

then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges



Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ b_{n+1} \leq b_n \qquad \text{and} \qquad \text{(ii)}\ \lim_{n\to\infty} b_n = 0$$

then $$|R_n| = |s - s_n| \leq b_{n+1}$$

- Important example

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

- Find a power series in terms of another one you know.

Examples $1/(1+8x^3)$, $x^2/(x+5)$

Examples $f(x)=\arctan(x)$ $f(x)=\ln(x)$

- Use the power series corresponding to $f'(x)$ to find the power series corresponding to $f(x)$.

- Use the power series of $\int f(x)dx$ to find the power series of $f(x)$

Examples $f(x)=1/(1+x)^2$

Example: Express $1/(1+x)^2$ as a power series and find radius of convergence?.

- Recall that the derivative and integral of a power series have the same interval of convergence.

- Use the alternate series estimation test to estimate the error when approximating a series by a partial sum.

Find the value of a certain integral below correct up to 8 decimal places

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23–26 Evaluate the indefinite integral as a power series. What is the radius of convergence?

23. $\displaystyle\int \frac{t}{1-t^8}\,dt$ **24.** $\displaystyle\int \frac{\ln(1-t)}{t}\,dt$

25. $\displaystyle\int \frac{x - \tan^{-1}x}{x^3}\,dx$ **26.** $\displaystyle\int \tan^{-1}(x^2)\,dx$

27–30 Use a power series to approximate the definite integral to six decimal places.

27. $\displaystyle\int_0^{0.2} \frac{1}{1+x^5}\,dx$ **28.** $\displaystyle\int_0^{0.4} \ln(1+x^4)\,dx$

29. $\displaystyle\int_0^{0.1} x\arctan(3x)\,dx$ **30.** $\displaystyle\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$