Compare the following two problems

Find a number x such that $x^2-3x+1=0$
In other words, find a number with certain properties.

Find a function f such that $f'(x)=3f(x)$ for all x in R.
In other words, find a function with certain properties.

$f'(x)=3f(x)$ is an example of a differential equation

A ***differential equation*** is an equation in which the unknown is a function and where one or more of the derivatives of this function appears. In other words, it is an equation that relates a function with one or more of its derivatives.

A ***solution*** of a differential equation is any function that when substituted for the unknown function f (f(x), y and f in the above examples) makes the equation an identity for all values of the variable (x or t in the above examples) in some interval.

$y' = 3x^2 y, \quad y = Ce^{x^3}$, C is a real number

$\frac{d^2 f}{dt^2} = -f, \quad y = a\cos(t) + b\sin(t)$, a and b are real numbers

The order of the highest order derivative of the unknown function is called the ***order*** of the differential equation.

$y' = y^2 + 1 + \sin(x), \quad order = 1,$
x is the independent variable
y=y(x) is the dependent variable

$\frac{d^2 f}{dt^2} + 3t\frac{df}{dt} = t^5 f \quad order = 2,$
t is the independent variable
f=f(t) is the dependent variable

Adding constrains to a differential equation

- Find a function f such that $f'(x)=3f(x)$ for all x in R and $f(1) = 3$
- Find a function f such that $f''(x)-5f'(x)+6f(x)=0$, $f(0)=1, f'(0)=-1$.

The constrains above are called e ***initial conditions*** of the differential equation. These are conditions that the solution and possibly some of its derivatives must satisfy.

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Example $f'(x)=3f(x)$, $f(0)=5$.

EXAMPLE

- Which of the following functions are solutions of the differential equation $y''+y=\sin(x)$?

$$a.\ y=\sin(x)$$
$$b.\ y=\cos(x)$$
$$c.\ y=x\sin(x)/2$$
$$d.\ y=-x\cos(x)/2$$

EXAMPLE: Solve the following differential equations

$$y'=3,\quad y(1)=4$$
$$y'=3+x,\quad y(1)=4$$
$$y'=y,\quad y(0)=2$$
$$y'=xy$$
$$xy'=0,\ y(0)=7$$
$$xy'=\frac{1}{x}y$$

We will study methods to solve some differential equations. For now, we will use trial and error.

Mathematical models

- The goal is not to produce an identical copy of the real object but give a representation of some aspect of the object.
- We can make a model by simplifying assumptions and combining aspects that may or may not belong together.
- Once the model is build, one should compare predictions of the model with data.

Modeling with differential equations

Quantities involved in a model
- independent variable (almost always time in our course)
- dependent variable (function of the independent variable)
- parameters (quantities which do not change with time. They can be adjusted).

Assumption: The number of individuals in a population grows at a rate proportional to the size of this population.

dP/dt = k P

When making a model one must:
1. State assumptions. This assumptions should describe relationships between quantities.
2. Describe variables and parameters used in the model.
3. Use assumptions in 1. to derive equations involving variables and parameters in 2. Key words are for instance: “rate of change of…” or “rate of increase of…”, “velocity”, “acceleration”, “proportional to”

Independent variable: t

Dependent Variable: P(t)

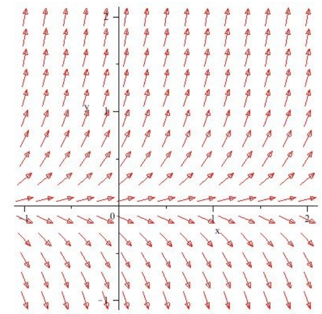
Parameter: k

An elementary model of population growth

Assumption: The number of individuals in a population grows at a rate proportional to the size of this population.

dP/dt = k P

This is another example of a differential equation.
It is a first order differential equation (only first derivatives)

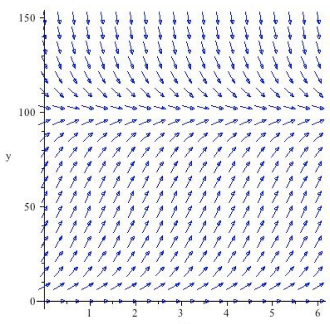


Each arrow represents the slope of the tangent line of the solution passing through that point

Another model of population growth

Assumption: The number of individuals in a population grows at a rate proportional to the size of this population when the number of individuals is small, but decreases when surpasses a certain number.

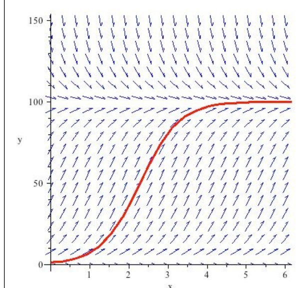
dP/dt = k P(1-P/M)



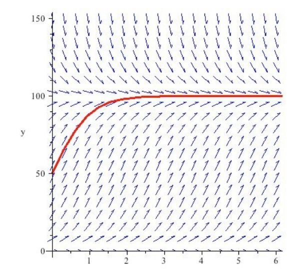
k=2 and M=100

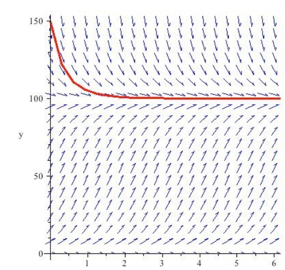
This equation “fits” the above assumption but it is not the only equation with that property. (The assumption does not give the decreasing rate)

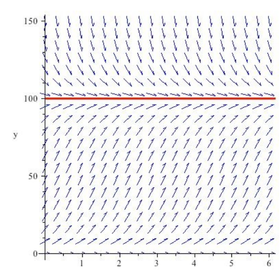
There are two constant solutions of this equations, P(t)=0 and P(t)=M for all t.

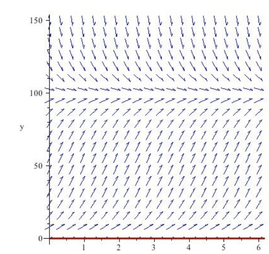


The curves represent solutions of the equation
dP/dt = 2 P(1 - P/100)
What are the initial conditions for each of them?

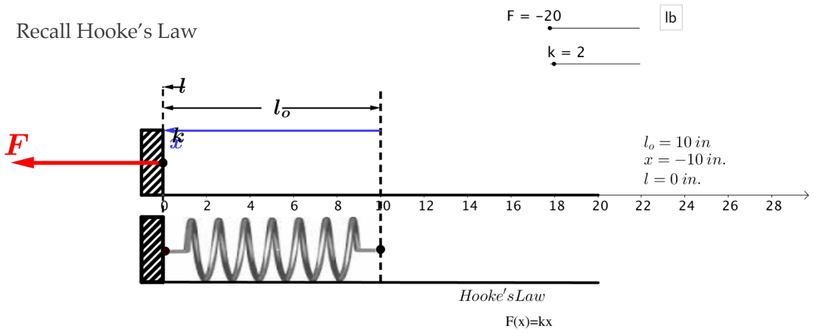








Recall Hooke's Law



The force that the spring exerts is

$F(x) = -kx$,

On the other hand, by Newton's Second Law,

$F(x) = m \cdot d^2x/dt^2$.

Then

$-k \cdot x = m \cdot d^2x/dt^2$.