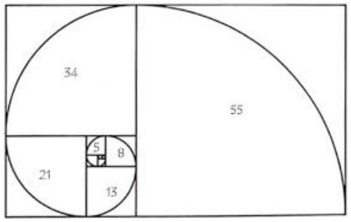
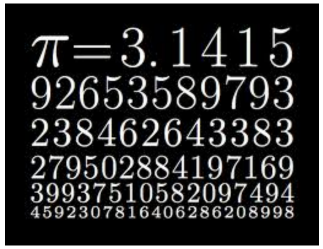
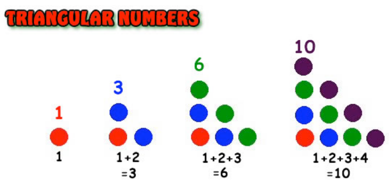
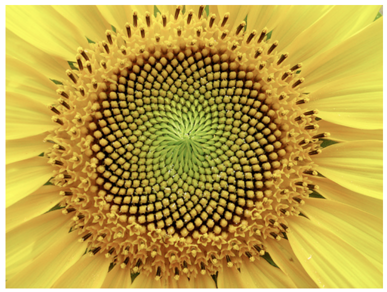




MAT 132 8.1 Sequences





An (infinite) sequence is a an infinite list of numbers written in order.

An (infinite) sequence is thus a function, where the domain is the set of positive integers and the range is the real numbers.

$$\{a_n\} = \{a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots\}$$

nth term

Examples
{1,1,1,1,..}
{1,2,3,..}
{½,-2/3,3/4,-4/5...}
{√2,√3,√4,... }
{1,4,1,5,9,2…}

In a sequence order matters and elements can be repeated.

Find a formula for the n-th term of each the above sequences.

A sequence is defined explicitly if there is a formula yields individual terms independently

Example: Consider the sequence of general term $a_n = 3^n$.

The first, second, third and fourth terms of this sequences are

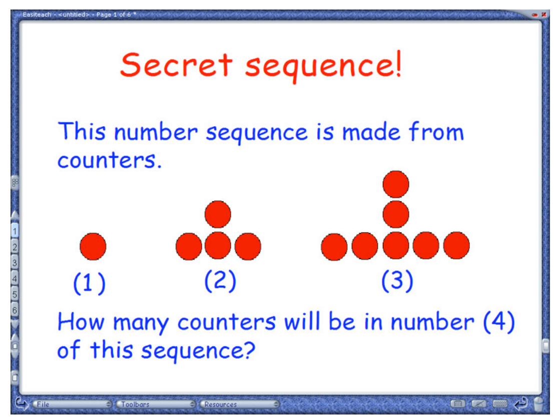
$$a_1 = 3^1 = 3,$$
$$a_2 = 3^2 = 9,$$
$$a_3 = 3^3 = 27,$$
$$a_4 = 3^4 = 81$$

Example: $$a_n = \frac{(-1)^n}{n^2+1}$$

To find the 100th term, plug 100 in for n:

$$a_{100} = \frac{(-1)^{100}}{100^2+1} = \frac{1}{10001}$$

Challenge: Find the 100-th term of the sequence below.



source: http://mrjosephsprecalculusblog.blogspot.com/

A sequence is defined recursively if there is a formula that relates a_n to previous terms.

Example 1: $b_1 = 4 \qquad b_n = b_{n-1} + 2 \quad \text{for all } n \geq 2$

Example 2: Fibonacci sequence $b_1=1, b_2=1, b_n=b_{n-1}+b_{n-2}$ for $n \geq 3$

Example 3: Collatz sequences

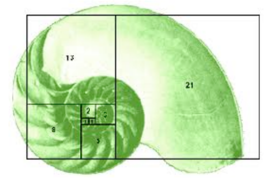
Example 1: $b_1 = 4$

$b_2 = b_1 + 2 = 6$

$b_3 = b_2 + 2 = 8$

$b_4 = b_3 + 2 = 10$

Can you give an explicit definition of the sequence in Example 1?



Fibonacci sequence in nature

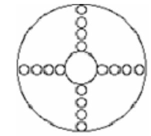
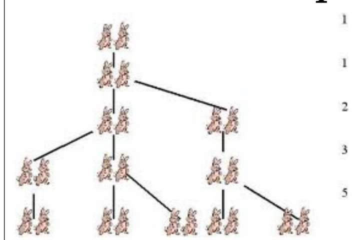
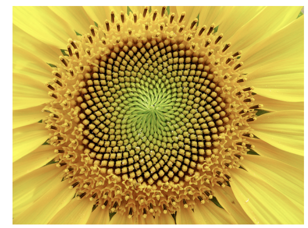


fig.1





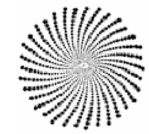
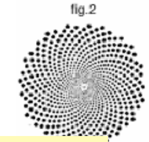


fig.2



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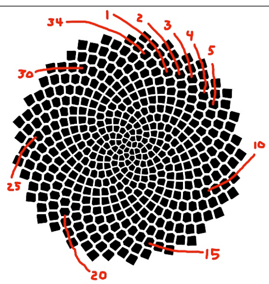
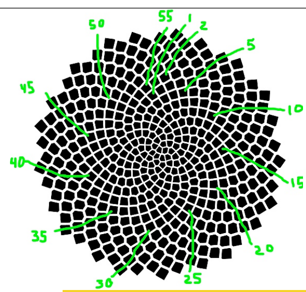
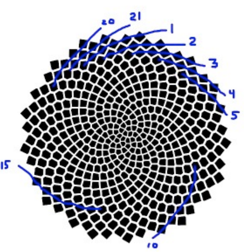
The golden angle, is 137.5 degrees.



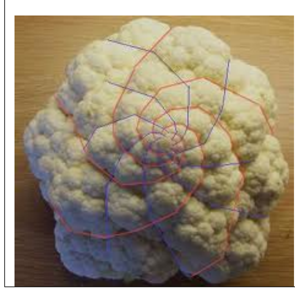


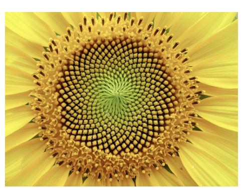


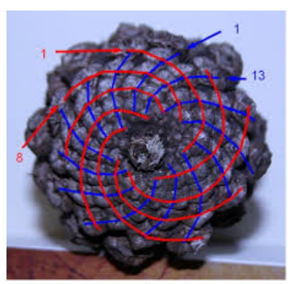
See more from Vi Hart "Open letter to Nickelodeaon"
https://www.youtube.com/watch?v=ahXIMUkSXX0

https://www.youtube.com/watch?v=gBxeju8dMho

1, 1, 2, 3, 5, 8, 13, **21**, **34**, **55**, 89...







Example of sequences defined recursively: Collatz sequences

f(n) = n/2 if n is even
3n+1 otherwise

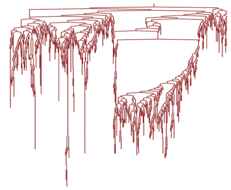
Start with a positive integer, say, 10,
$a_1=10$
$a_2=f(a_1)=5$
$a_3=f(a_2)=16$
and so on.

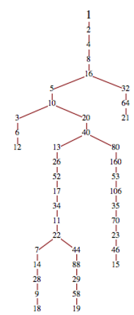
Conjecture:
No matter which number you start from, the sequence always reaches 1

This gives a recursively defined sequence for each "starting number", which seems to end in 1, 1,1,1.. for all starting numbers.
(Starting at a different number, you'll obtained a different sequence)

2011: The Collatz algorithm has been tested and found to always reach 1 for all numbers up to $5 . 7 \times 10^{18}$

Example of sequences defined recursively: Collatz sequences





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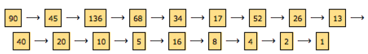
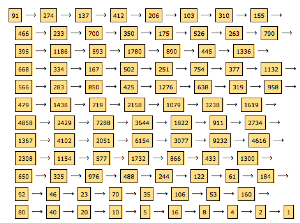




An arithmetic sequence is a sequence such that the difference between consecutive terms is constant.

Examples: $-5, -2, 1, 4, 7, \ldots$ $\quad d=3$

$\ln 2, \ln 6, \ln 18, \ln 54, \ldots$ $\quad d=\ln 6-\ln 2 = \ln\frac{6}{2} = \ln 3$

Arithmetic sequences can be defined recursively: $a_n = a_{n-1}+d$

or explicitly: $a_n = a_1 + d(n-1)$

A sequence is geometric if the quotient of consecutive terms is constant. That is consecutive terms have the same ratio.

Example: $1, -2, 4, -8, 16, \ldots$ $\quad r=-2$

$10^{-2}, 10^{-1}, 1, 10, \ldots$ $\quad r=\frac{10^{-1}}{10^{-2}} = 10$

Geometric sequences can be defined recursively: $a_n = a_{n-1} . r$

or explicitly: $a_n = a_1 . r^{n-1}$

A sequence is defined explicitly if there is a formula that allows you to find individual terms independently.

Ex: $a_n=n/(n^2+1)$

Any real-valued function defined on the positive real yields a sequence (explicitly defined).

Example: $f(x)=(x+2)^{½}$
n-th of the sequence: $a_n=(n+2)^{½}$

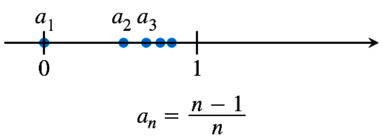
A sequence is defined recursively if there is a formula that relates a_n to previous terms.

An arithmetic sequence has a common difference between terms.

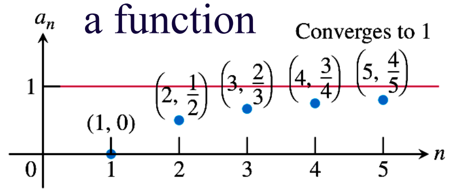
An geometric sequence has a common ratio between terms.

Write the first terms of the sequence $a_n = \dfrac{n-1}{n}$

Plot these terms on a number line

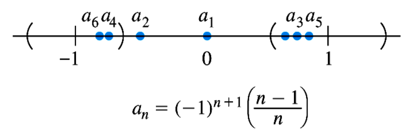


Plot the sequence as a function

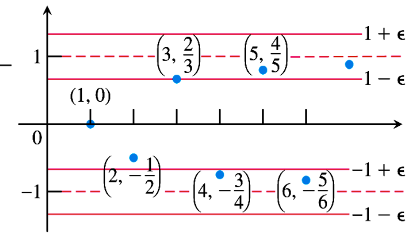


The terms in this sequence get closer and closer to 1.
The sequence CONVERGES to 1.

Consider the sequence $a_n = \dfrac{(-1)^{n+1}(n-1)}{n}$



Neither the ϵ-interval about 1 nor the ϵ-interval about −1 contains all a_n satisfying $n \geq N$ for some N.



The terms in this sequence do not get close to any (single) number when n grows.

The sequence $\{a_n\}$ *converges to L* if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.

We write $\lim_{n \to \infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

Example

$$a_n = \frac{n-1}{n}$$

Otherwise, that is if $\{a_n\}$ does not converges to any number, we say that $\{a_n\}$ *diverges*.

Example

$$a_n = \frac{(-1)^{n+1}(n-1)}{n}$$

Recall:
A sequence is geometric if the quotient of consecutive terms is constant. That is consecutive terms have the same ratio.

Example: $1, -2, 4, -8, 16, \ldots$ $r = -2$

$10^{-2}, 10^{-1}, 1, 10, \ldots$ $r = \dfrac{10^{-1}}{10^{-2}} = 10$

Geometric sequences can be defined recursively: $a_n = a_{n-1} \cdot r$

or explicitly: $a_n = a_1 \cdot r^{n-1}$

Can you find examples of convergent geometric sequence? And of diverent geometric sequences?

Determine whether the sequences below are convergent.

1. $a_n = 3^n$,
2. $a_n = (½)^n$
3. $a_n = (-1)^n$
4. $a_n = (-2)^n$
5. $a_n = (-0.1)^n$
6. $a_n = (3/2)^n$

7 The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

2 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

- Examples: Study whether the sequences below converge using the theorem above (if possible)

$$a_n = \frac{n-1}{n}$$

$$a_n = \frac{(-1)^{n+1}(n-1)}{n}$$

- Example: The above theorem cannot be used to prove that the sequence $a_n=1/n!$ converges. Why?

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Example: Below is the n-th term of some sequences
Determine whether the corresponding sequences converge and if so, find the limit.

1. $a_n = 1/n$
2. $a_n = 1/n + 3(n+1)/n^2$
3. $b_n = (a_n)^2$ (a_n as in 2.).
4. $a_n = n!/(n+1)!$
5. $a_n = (n+1)!/n!$
6. $a_n = 1/\ln(n)$.
7. $a_n = n/\ln(n)$.
8. $a_n = n \,.\, \sin(1/n)$.

The squeeze theorem

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

- Example: Use the "squeeze theorem" above to determine whether the sequence $\{a_n\}$ defined by $a_n = (n^2+1)/n^3$ is converges and if so, find the limit.

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Give examples of
1. Increasing, convergent sequences.
2. Decreasing convergent sequences.
3. Increasing divergent sequences.
4. Decreasing divergent sequences.
5. Convergent sequences that are not increasing and not decreasing
6. Divergent sequences that are not increasing and not decreasing

List of Types of sequences and examples

Defined explicitly Ex: $a_n=n/(n^2+1)$
Defined recursively Ex: $a_1=1$, $a_2=1$, $a_n=a_{n-1}+a_{n-2}$, $n\geq 3$.
Defined by function
Example: $f(x)=(x+2)^{½}$ $a_n=(n+2)^{½}$

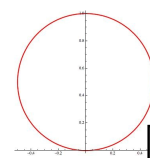
Convergent $a_n=1/n$
Divergent, $a_n=n$ or $a_n=(-1)^n$

Arithmetic $a_n=a_1+(n-1)\,.\,d$
Geometric $a_n=a_1\,.\,r^{n-1}$

Increasing $a_n=(1-1/n)$
Decreasing $a_n=1/n$
Bounded $a_n=\sin(n)$

"Tricks" to determine whether sequences are convergent or divergent and to find the limit if they are convergent.
- Squeeze theorem
- L'Hopital

Example: Compute the arc length of the circle r = sinϴ
ϴ in [0,2π]

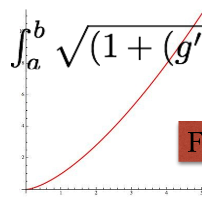


POLAR

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example: Compute the arc length the graph of the curve $y=x^{3/2}$,
x in [0,5]

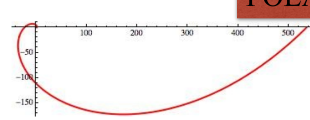
$$\int_a^b \sqrt{(1 + (g'(x))^2}\ dx$$



FUNCTION

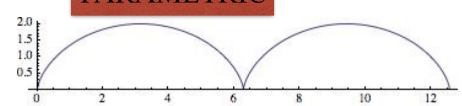
Example: Compute the length arc of the spiral $r = e^ϴ$
in [0,2π]

POLAR



Example: Compute the length two arcs of the cycloid x= ϴ - sinϴ, y=1-cos ϴ. (Hint (1-cos(t))=2 sin²(t/2))

PARAMETRIC



$$\int_c^d \sqrt{((f'(t))^2 + (g'(t))^2}\ dt$$