Summary on solving the linear second order homogeneous differential equation

To find the general solution of the differential equation **A** y'' + **B** y' + **C** y = 0 we consider the characteristic equation:

$$A x^2 + B x + C = 0$$

Set $\Delta = B^2 - 4AC$.

Example

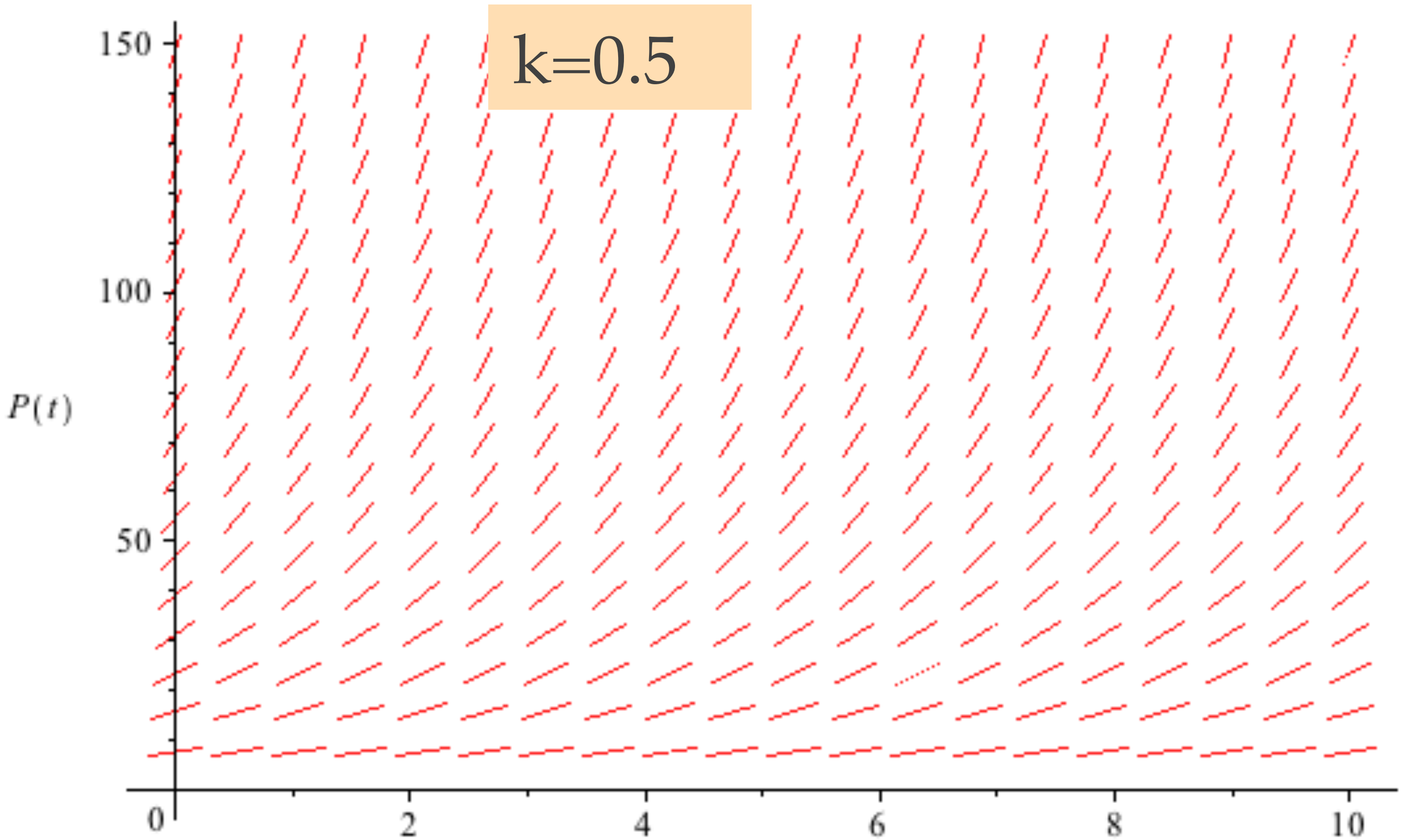
1. Solve the initial-value problem y'' + 2 y' + y=0, y(0)=1, y(1)=3.
2. 2y''+5y+3y=0, y(0)=3, y'(0)=-4.

	roots of the characteristic polynomial	General solution
$\Delta>0$	two distinct real roots r	c_1
$\Delta<0$	two complex roots $\alpha+i\beta$ and $\alpha - i\beta$	e^{α} $\sin(\beta x))$
$\Delta=0$	one double real root r	c_1

The Logistic Equation

An elementary model of population growth

Assumption: The number of individuals in a population grows at a rate proportional to the size of this population.

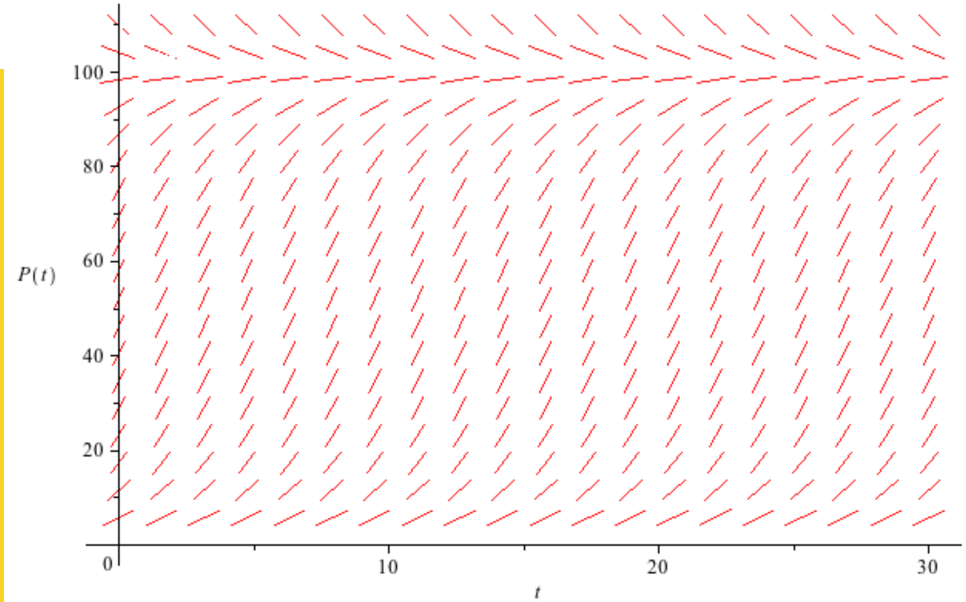


$dP/dt = k\,P$

The direction of each line represents the slope of the tangent line of the solution passing through the point (t, P(t))

Another model of population growth

Assumption: The number of individuals in a population grows at a rate proportional to the size of this population when the number of individuals is small, but decreases when surpasses a certain number.



k=0.5 and M=100

Logistic differential equation
dP/dt = k P(1-P/M)

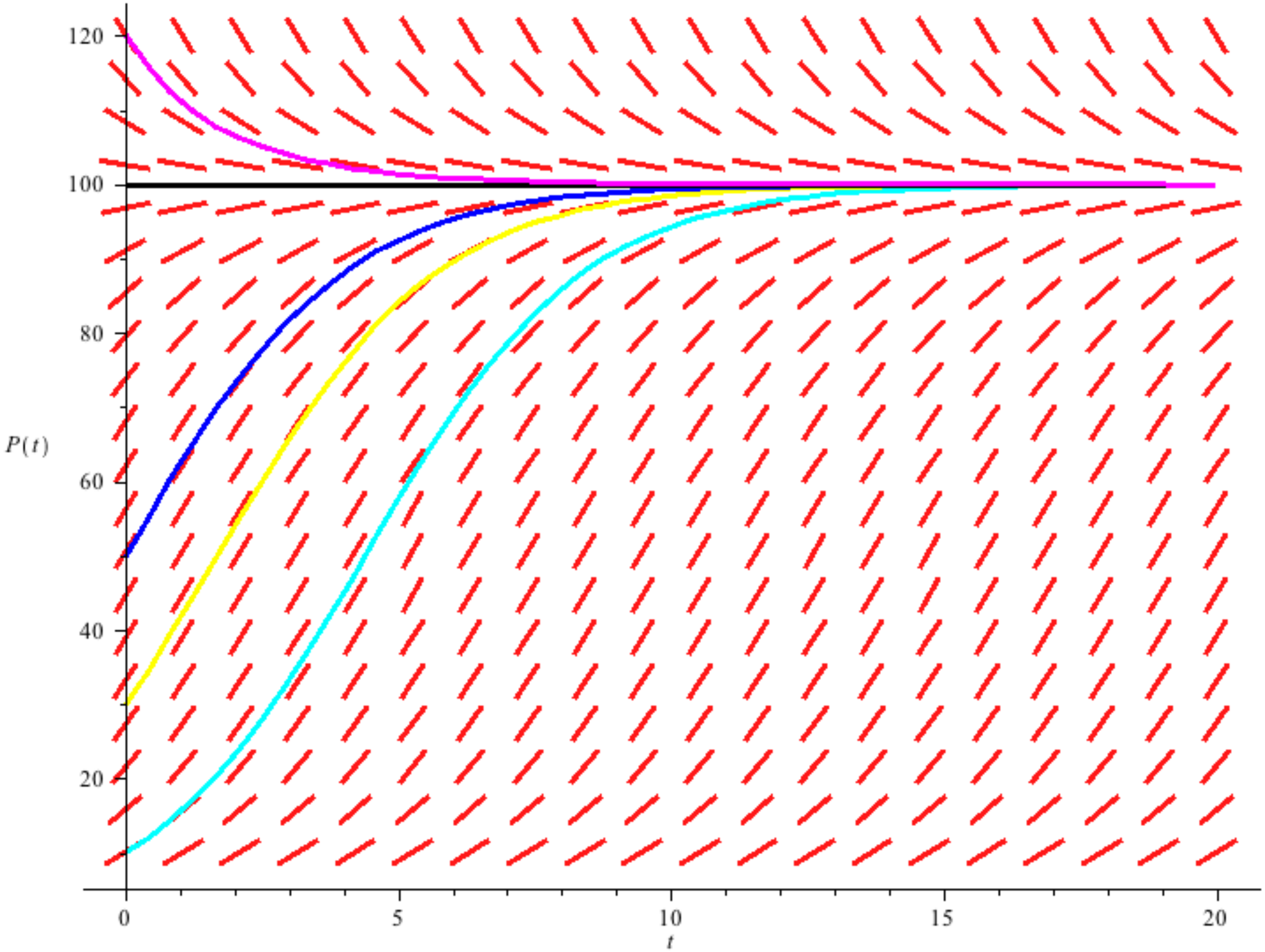
M and k are parameters which depend on the population.

There are two constant solutions of this equations, P(t)=0 and P(t)=M

M is the ***carrying capacity***, the amount that when exceeded will result in the population decreasing.

The direction field of the logistic differential equation and some solutions. (M=100, k=0.5)

What are the initial conditions of the plotted solutions?



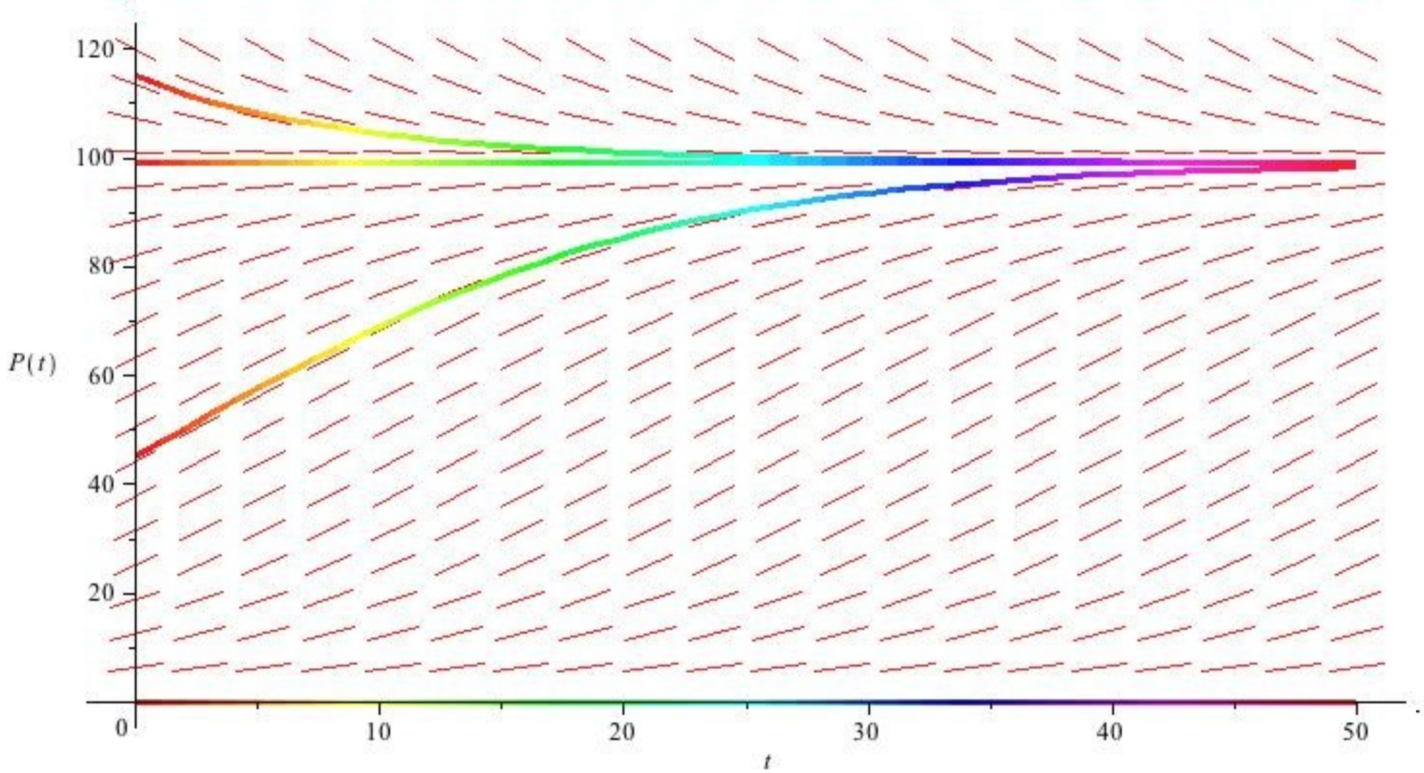
t = time

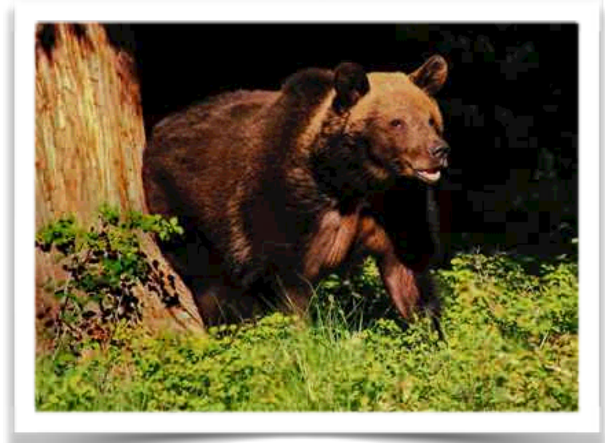
P (or $P(t)$) = the population size at time t.

Logistic differential equation

$dP/dt = k\,P(1-P/M)$

1. Solve the initial value problem for logistic differential equation with initial condition $P(0)=P_0$.
2. Study what happens with each solution when time goes to infinity.





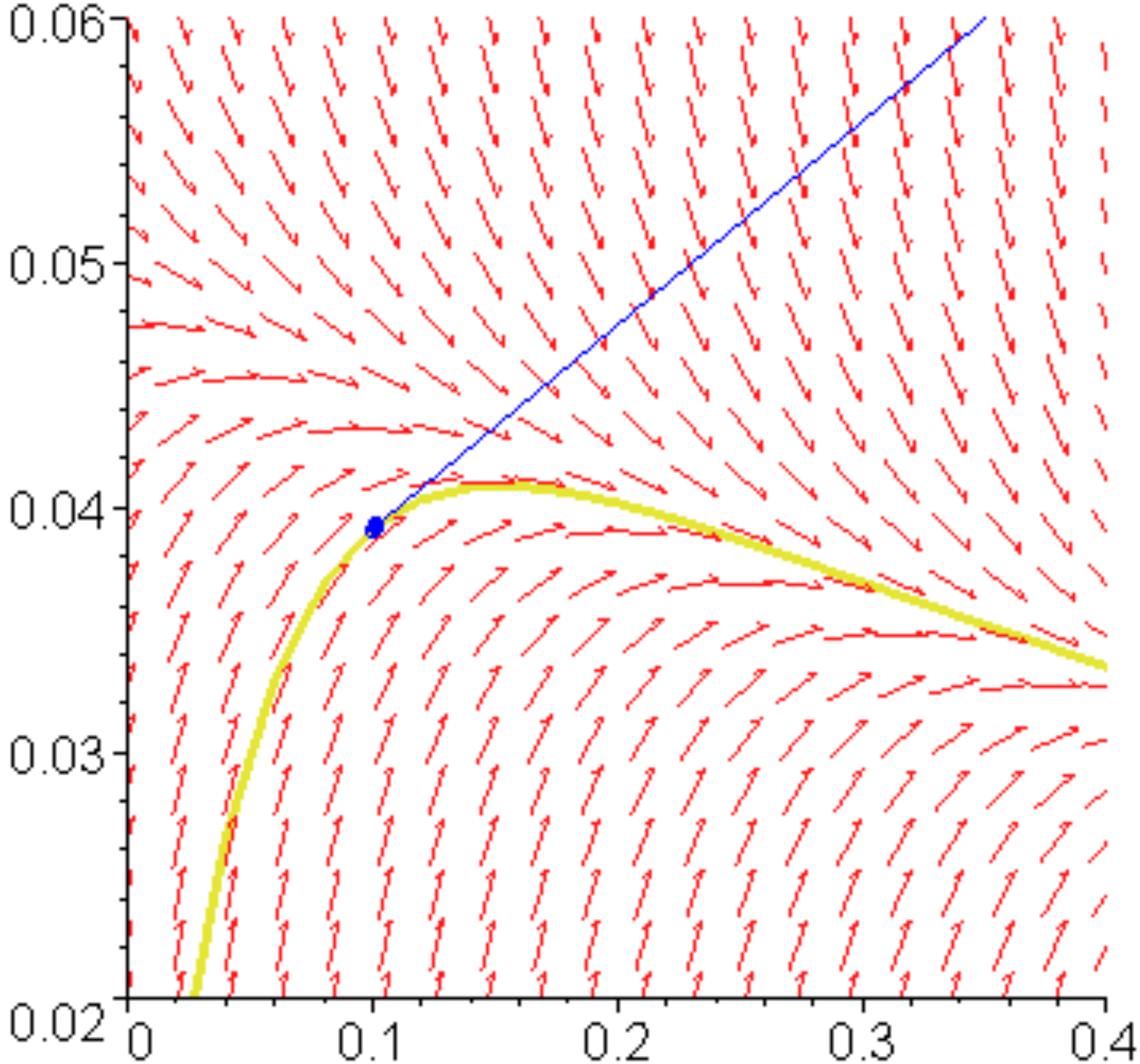
Eleven grizzly bears were introduced to a national park a few years ago. The relative growth rate for grizzly bears is 0.1 and the park can support a maximum of 99 bears. (This is not a realistic relative growth)

1. Assuming a logistic growth model, will the bear population reach 5? 70? 100? 120? 99?
2. Use Euler's method with step size 2 to estimate the number of bears after 4 years.
3. Use Euler's method with step size 2 to estimate the number of bears after 4 years assuming the initial number of bears is 44.
4. Find an explicit solution of the corresponding differential equation and check the accuracy of your estimations in 2..

Example from
http://www.ece.uwaterloo.ca/~ece104/

Euler’s method

$y'(t) = -21\, y(t) + e^{-t}$
$y(0) = 0$



9. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.

(a) Write a differential equation that is satisfied by y.

(b) Solve the differential equation.

(c) A small town has 1000 inhabitants. At 8 AM, 80 people have heard a rumor. By noon half the town has heard it. At what time will 90% of the population have heard the rumor?

Suppose a species of fish in lake is modeled by a logistic population model with relative growth rate of $k = 0.02$ per year and carrying capacity of $K = 50$.

a. Write the differential equation describing the logistic population model for this problem.

b. Draw a vector field for this problem.

c. Determine the equilibrium solutions for this model.

d. If 25 fishes are introduced in the lake, estimate the time it will take to have 8000 fish in the lake.

6. The table gives the number of yeast cells in a new laboratory culture.

Time (hours)	Yeast cells	Time (hours)	Yeast cells
0	18	10	509
2	39	12	597
4	80	14	640
6	171	16	664
8	336	18	672

(a) Plot the data and use the plot to estimate the carrying capacity for the yeast population.
(b) Use the data to estimate the initial relative growth rate.
(c) Find both an exponential model and a logistic model for these data.
(d) Compare the predicted values with the observed values, both in a table and with graphs. Comment on how well your models fit the data.
(e) Use your logistic model to estimate the number of yeast cells after 7 hours.